# Numerical solution of differential equations Numerical Techniques and programming in MATLAB

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# Initial value problems:



We will focus on the following initial value problem: Find the function $y(x)$ that satisfies

$$\left.\begin{aligned} &\frac{dy}{dx} = f(x, y), \quad a \leq x \leq b \\ &y(a) = \alpha \end{aligned}\right\} \tag{1}$$

Remark: If we know how to deal with first order differential equations, then we can also attempt to solve nth order differential equations. Because it is possible to write nth order differential equations into a set of n first order differential equations.

# Uniqueness of the solution:



We know that under some restrictions on the given function $f$, Picard's Theorem guarantee existence of unique local solution. But we are interested in uniqueness of the global solution ( the solution where the differential equation is defined ) Let us recall the following theorem:

**Theorem**

- *Let* $D := \{(x, y) | a \leq x \leq b, -\infty \leq y \leq \infty\}$.
- $f$ *is continuous on* $D$.
- $f$ *is Lipschitz continuous in* $y$ *on* $D$.

*OR*

$\frac{\partial f}{\partial y}$ *is continuous on* $D$.

$\Rightarrow$ (1) *has a unique solution for* $a \leq x \leq b$.

Recall: $f(x, y)$ is said to be Lipschitz continuous in $y$ on a set $D \subset \mathbb{R}^2$ if $\exists L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

whenever $(x, y_1), (x, y_2) \in D$.

Ex. $f(x, y) = |y|$, $D := \{(x, y)|\,|x| \leq 1,\ |y| \leq 1\}$, $f$ is Lipschitz continuous on $D$ with $L = 1$.

Ex. $f(x, y) = \sqrt{y}$, $D := \{(x, y)|\,|x| \leq 1,\ |y| \leq 1\}$, $f$ is NOT Lipschitz continuous on $D$.
Taking $y_1 = 0$ and $y_2$ as arbitrary, then

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{\sqrt{y_1}}{y_1} = \frac{1}{\sqrt{y_1}} \to \infty$$

as $y_1 \to 0$.

Remark: Of course, Lipschitz continuity is a mild condition in comparison to $\dfrac{\partial f}{\partial y}$:

Consider $f(x, y) = |y|$, $D := \{(x, y) | \, |x| \leq 1, \, |y| \leq 1\}$, $\dfrac{\partial f}{\partial y}$ does not exist at $(0, 0)$. However, $|y|$ is Lip. continuous with Lip. constant $L = 1$ (can you see this?)

- Well-posed problem: We say (1) is well-posed if
  - (1) has a unique solution,
  - solution depends on the given data, i.e., $f$ and $\alpha$.
- Remark: Under the conditions of Theorem 1, (1) is well-posed.
- Remark: It would be difficult to find an analytic/exact solution of (1) in a closed form. Therefore, we look for suitable numerical scheme which provide an approximate solution.
- Idea: Discretize the domain (interval) and the given equation.

- Notations:
  - $y_i = y(x_i)$: exact solution at point $x_i$.
  - $w_i = w(x_i)$: numerical solution at point $x_i$.

  We write $w_i \approx y_i = y(x_i)$.
- Error:
  1. How well numerical scheme approximates (1)?
  2. How well solution of numerical scheme $(w_i)$ approximates the solution of (1)

$$\max_{1 \leq i \leq N} |y_i - w_i| = ?$$

# Some numerical schemes/methods



- Recall: Our differential equation

$$\left.\begin{array}{l} \dfrac{dy}{dx} = f(x,y), \quad a \leq x \leq b \\ y(a) = \alpha \end{array}\right\}, \tag{2}$$

Discretize the interval as $x_i = x_0 + ih, x_0 = a$. Integrate (2) from $x_0$ to $x$.

$$\begin{aligned} \int_{x_0}^{x} \frac{dy}{dx} dx &= \int_{x_0}^{x} f(t, y(t)) dt \\ \Rightarrow y(x) &= \alpha + \int_{x_0}^{x} f(t, y(t)) dt. \end{aligned} \tag{3}$$

- Remark: (2) and (3) are equivalent.

- Basic idea: Using $y_0$, we find $y_1$, then $y_2$, .. and so on.
- First approach: Approximate the integral involved in (3) (but we don't know $y(x)$).
- Second approach: Approximate the derivatives involved in (2).
- Recall:
  - $\int_a^b f(x) = (b-a)f(a)$.
  - $\int_a^b f(x) = \frac{(b-a)}{2}[f(a)+f(b)]$.
  - $\int_a^b f(x) = \frac{(b-a)}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]$.

We write

- Method I: $\int_{x_0}^{x_1} f(t,y(t))dt = hf(x_0,y(x_0)) = hf(x_0,y_0)$
- Method II: $\int_{x_0}^{x_1} f(t,y(t))dt = \frac{h}{2}[f(x_0,y_0)+f(x_1,y_1)]$
- Method III:

$$\int_{x_0}^{x_1} f(t,y(t))dt = \frac{h}{6}\left[f(x_0,y_0)+4f(x_{1/2},y_{1/2})+f(x_1,y_1)\right],$$

where $x_{1/2} = \dfrac{x_0+x_1}{2}$ and $y_{1/2} = \dfrac{y_0+y_1}{2}$.

Using these methods we can compute $w_1$ with the help of $w_0 = \alpha$ in the following manner:

- Method I: $w_1 = w_0 + hf(x_0, w_0)$
- Method II: $w_1 = w_0 + \frac{h}{2}[f(x_0, w_0) + f(x_1, z_1)]$ (Use method I to compute $w_1$),i.e., $z_1 = w_0 + hf(x_0, w_0)$
- Method III:

$$w_1 = \frac{h}{6}\left[f(x_0, w_0) + 4f(x_{1/2}, w_{1/2}) + f(w_1, w_1)\right],$$

. Problem is we must have $w_{1/2}$.

# Euler's Method



We have the following schemes:

- Scheme 1: $w_{i+1} = w_i + hf(x_i, w_i)$, $i = 0, 1, 2, ..$ (Euler's method).
- Scheme 2: $w_{i+1} = w_i + \frac{h}{2}[f(x_i, w_i) + f(x_{i+1}, w_{i+1})]$ ( Modified Euler's method).

Here, $w_{i+1}$ is unknown.

Idea: Use Euler's method to compute $w_{i+1}$. Write

$$w_{i+1} = w_i + \frac{h}{2}[f(x_i, w_i) + f(x_{i+1}, z_{i+1})]$$

where $z_{i+1} = w_i + hf(x_i, w_i)$.

Remark: Method III leads to Runge-Kutta method. After having appropriate approximation of $w_{1/2}$.

- Scheme 3:

$$w_{i+1} = w_i + \frac{1}{6}\left[m_1 + 2m_2 + 2m_3 + m_4\right],$$

where

- $m_1 = hf(x_i, w_i)$
- $m_2 = hf(x_i + \frac{h}{2}, w_i + \frac{m_1}{2})$
- $m_3 = hf(x_i + \frac{h}{2}, w_i + \frac{m_2}{2})$
- $m_4 = hf(x_i + h, w_i + m_3)$

This method is called Runge-Kutta method. One of the best method for solving differential equations.

$$w_{i+1} = w_i + \left[ \frac{h}{4} f(x_i, w_i) + \frac{3h}{4} f(x_i + \frac{2h}{3}, \tilde{w}_i) \right],$$

where

$$\tilde{w}_i = w_i + \frac{2h}{3} f(x_i, w_i)$$

This is called optimal RK-2.

# Schemes based on second approach (Numerical differentiation) :



Taylor's series:

$$
\begin{aligned}
f(x) &= f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + .. \\
&+ \frac{(x - x_0)^n}{n!} f^n(x_0) + E
\end{aligned}
$$

$E = \dfrac{(x - x_0)^{n+1}}{(n+1)!} f^{n+1}(\xi)$, for some $\xi$ lying between $x_0$ and $x$.

Problem: We do not know $\xi$, hence it would be difficult to compute error with this formula. However, this can be used in order to find the upper bound of the error, i,e., we have a number in hand such that error should not exceed by this number.

Assume that the solution of (1) is smooth ( have continuous derivative of order $n+1$ on the interval $(a,b)$: Then

$$\begin{aligned} y(x) &= y(x_i) + (x-x_i)y'(x_i) + \frac{(x-x_i)^2}{2!}y''(x_i) + \ldots \\ &\quad + \frac{(x-x_i)^n}{n!}y^n(x_i) + Error \end{aligned} \tag{4}$$

$Error = \dfrac{(x-x_i)^{n+1}}{(n+1)!}y^{n+1}(\xi)$, for some $\xi$ lying between $x_i$ and $x$.

Take $x = x_{i+1}$ in (4), then we have

$$y_{i+1} = y_i + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \dots + \frac{h^n}{n!}y^n(x_i) + \frac{h^{n+1}}{(n+1)!}y^{n+1}(\xi) \quad (5)$$

for some $\xi$ lying between $x_i$ and $x_{i+1}$.

Idea: Since $h$ is small $h^k \to 0$, $k > 0$, we truncate this series after some terms. This would leads to different scheme. we expect that having more number of terms would provide more accurate solution. Let us try !

- Method 1. Take $n = 1$,

$$\begin{aligned} y_{i+1} &= y_i + hy'(x_i) + \frac{h^2}{2!}y''(\xi) \\ &= y_i + hf(x_i, y_i) + \frac{h^2}{2!}y''(\xi). \end{aligned}$$

  Therefore, the scheme is $w_{i+1} = w_i + hf(x_i, w_i)$. Same is Euler Method. ( did you get it?).

- Method 2. Take $n = 2$,

$$\begin{aligned} y_{i+1} &= y_i + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(\xi) \\ &= y_i + hf(x_i, y_i) + \frac{h^2}{2!}\frac{d}{dx}f(x, y)|_{x=x_i} + \frac{h^3}{3!}y'''(\xi) \end{aligned}$$

  Scheme is $w_{i+1} = w_i + hf(x_i, w_i) + \frac{h^2}{2!}\frac{d}{dx}f(x_i, w_i)$.

We have used $y' = \frac{dy}{dx} = f(x, y)$

Similarly, we have

- Method 3. For $n = 3$, scheme is

$$w_{i+1} = w_i + hf(x_i, w_i) + \frac{h^2}{2!}\frac{d}{dx}f(x_i, w_i) + \frac{h^3}{3!}\frac{d^2}{dx^2}f(x_i, w_i).$$

- Method 4. For $n = 4$, scheme is

$$\begin{aligned} w_{i+1} &= w_i + hf(x_i, w_i) + \frac{h^2}{2!}\frac{d}{dx}f(x_i, w_i) + \frac{h^3}{3!}\frac{d^2}{dx^2}f(x_i, w_i) \\ &+ \frac{h^4}{4!}\frac{d^3}{dx^3}f(x_i, w_i). \end{aligned}$$

## Example

$$\frac{dy}{dx} = y - x^2 + 1, \; 0 \leq x \leq 2$$
$$y(0) = 0.5, \; h = 0.2$$

find $y(0.6)$.

**Soln:** We have $x_1 = 0.2, x_2 = 0.4, x_3 = 0.6$. Euler's scheme:

$$w_{i+1} = w_i + hf(x_i, w_i).$$

We want to find $w_3$. For
$i = 0, \; w_1 = w_0 + hf(x_0, w_0) = 0.5 + 0.2(0.5^2 - 0 + 1) = 0.8$

$$w_2 = w_1 + hf(x_1, w_1) = 1.15, \quad w_3 = w_2 + hf(x_2, w_2) = 1.55.$$

Euler's modified method:

$$\begin{aligned} w_1 &= w_0 + \frac{h}{2}\left[f(x_0, w_0) + f(x_1, z_1)\right] \\ z_1 &= w_0 + hf(x_0, w_0) = 0.5, \\ w_1 &= 0.5\frac{h}{2}\left[(w_0^2 - x_0^2 + 1) + (z_1^2 - x_1^2 + 1)\right] \end{aligned}$$

Similarly, we can find $w_2$ and $w_3$.

## Using Taylor's series:

- $n = 1 \Rightarrow$ Euler method.
- $n = 2$,

$$w_{i+1} = w_i + hf(x_i, w_i) + \frac{h^2}{2!}\frac{d}{dx}f(x_i, w_i)$$

$$\begin{aligned}
f(x,y) &= y - x^2 + 1 \\
\frac{df}{dx} &= \frac{dy}{dx} - 2x + 1 = f(x,y) - 2x + 1 \\
&= (y - x^2 + 1) - 2x + 1 \\
\frac{d^2f}{dx^2} &= f(x,y) - 2x - 2 = (y - x^2 + 1) - 2x - 2
\end{aligned}$$

$$w_1 = w_0 + hf(x_0, w_0) + \frac{h^2}{2!}[(w_0 + x_0^2 + 1) - 2x_0 + 1]$$

$w_1 = 0.83, w_2 = 1.21, w_3 = 1.65$. For $n = 4$,
$w_1 = 0.82, w_2 = 1.21, w_3 = 1.64$.

- Major disadvantages of Taylor's Series method of higher order is the evaluation of the derivatives.
- Question: What are the advantages?
- Answer: We will see in moment.

# Error



Local truncation error: The amount by which exact solution satisfies the numerical scheme.

One step method: $\dfrac{w_{i+1} - w_i}{h} = \phi(f, x_i, w_i, h)$

Euler's method: $\dfrac{y_{i+1} - y_i}{h} = f(x_i, y_i) + \frac{h}{2} f'(\xi, y(\xi)),$

$$\tau_i = hf'(\xi, y(\xi))$$
$$\tau_i = hy''(\xi)$$

Taylor's series $\tau_i = \mathcal{O}(h^n)$.

Consistent: If $\tau_i \to 0$ as $h \to 0$.

Convergence: $\lim_{h\to 0} \max_{1\le i\le N} |y_i - w_i| = 0.$

Stability: Let $\tilde{w}_i$ be the solution of

$$\begin{aligned} \frac{dy}{dx} &= f(x,y),\ a \le x \le b \\ y(a) &= \tilde{\alpha}. \end{aligned}$$

Then $|\tilde{w}_i - w_i| \le K(x_i)|\tilde{\alpha} - \alpha|$.

Order of convergence: If $\tau_i$ is $\mathcal{O}(h^p)$. Then the scheme has $p$ order of convergence.

- We look for high order of convergence.
- Note that we can increase order of convergence of the Taylor series method by adding more numbers of terms in the Taylor series.
- This can be think of an advantage of Taylor series method.

**Theorem**

*Convergence* $\Leftrightarrow$ *Consistency* $+$ *Stability.*

## Multistep method:

- We have seen, to compute $w_{i+1}$, we use only $w_i$. In particular, to find $w_1$ we need only $w_0$.
- If we need $w_{i-2}, w_{i-1}, w_i$ to compute $w_{i+1}$, then this method is called multistep method. Here 3-step method.

**Derivation of multi-step method:** $\frac{dy}{dx} = f(x, y)$.

Integrate this equation from $x_i$ to $x_{i+1}$

$$\int_{x_i}^{x_{i+1}} \frac{dy}{dx} = \int_{x_i}^{x_{i+1}} f(x, y(x))dx \qquad (6)$$

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} f(x, y(x))dx. \qquad (7)$$

Idea: $f(x, y(x)) \approx P_m$

Two step method: To compute $w_{i+1}$, we need $w_i, w_{i+1}$

$$f(x, y(x)) \approx P_1$$

Write $P_1$ by using $x_i$ and $x_{i-1}$.

$$P_1 = \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i, y_i) + \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}, y_{i-1}) + \frac{5h^3}{12} y'''(\xi)$$

$$\int_{x_i}^{x_{i+1}} P_1 dx = \frac{3h}{2} f(x_i, y_i) + (\frac{-h}{2}) f(x_{i-1}, y_{i-1})$$

Scheme:

$$w_{i+1} = w_i + \frac{3h}{2} f(x_i, w_i) - \frac{h}{2} f(x_{i-1}, w_{i-1})$$
$$\tau_i = \mathcal{O}(h^2)$$

Two step Adams-Bashforth method:

$$w_2 = w_1 + \frac{3h}{2} f(x_1, w_1) - \frac{h}{2} f(x_0, w_0)$$

To compute $w_1$, one can use any method for which $\tau_i = \mathcal{O}(h^2)$. (Second order Taylor's series method)

It is very hard to say which method is the best, it depends what you want? Best method for us would

- Esay to implement
- High order convergence.

There is no RAMA's ARROW which can solve any differential equations

Thank you for your attention!