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Tutorials-4

Topic: Applications of Laplace and Fourier transforms over PDEs :

Applications of Laplace transforms to solve Heat, Wave and Laplace equations, applications of Fourier transforms, Fourier sine and Fourier cosine transforms to solve Heat, wave and Laplace equations.

1. Find the bounded solution of $\partial^2 y / \partial x^2 - \partial^2 y / \partial t^2 = xt$, when $y = 0 = \partial y / \partial t$ at $t = 0$ and $y(0, t) = 0$
2. Solve $\partial v / \partial t = 2 (\partial^2 y / \partial x^2)$ where $y(0, t) = y(5, t) = 0$ and $y(x, 0) = 0 \sin 4\pi x$
3. (a) Find the bounded solution of $\partial y / \partial t = \partial^2 y / \partial x^2$ $x > 0, t > 0$ given that $y(0, t) = 1$ and $y(x, 0) = 0$.
(b) Solve $\partial^2 \frac{y}{\partial t^2} = 9 \left(\frac{\partial^2 y}{\partial x^2} \right)$, where $y(0, t) = 0, y(2, t) = 0$ and $y(x, 0) = 20 \sin 2\pi x, t_t(x, 0) = 0$
4. Solve $\partial y / \partial t = 3 (\partial^2 y / \partial x^2)$, $y_s(0, t) = 0, y(\pi/2, t) = 0$ and $y(x, 0) = 20 \cos 3x - 5 \cos 9x$.
5. Solve the boundary value problem $\partial u / \partial x + 4(\partial u / \partial t) = -8t, t > 0 > x > 0$, with the conditions $u = 0$, when $t = 0, x > 0$; $u = 2t^2$, when $x = 0, t > 0$.
6. Solve the boundary value problem $\partial^2 u / \partial t^2 = a^2 (\partial^2 u / \partial x^2), t > 0, x > 0$ Where $u(x, 0)x \geq 0, u_t(x, 0) = 0, x > 0, u(0, t) = t, \lim_{x \rightarrow \infty} u(x, t) = 0, t \geq 0$.
7. Solve the boundary value problem, $\partial^2 u / \partial x^2 = (l/c^2) (\partial^2 u / \partial t^2), t > 0, x > 0$ with the conditions (i) $u = E \sin at$, when $x = 0, t > 0$ (ii) u is finite when $x \rightarrow \infty$ (iii) $u = \partial u / \partial t = 0$ when $t = 0, x > 0$.
8. Solve the following boundary value problem $\partial^2 u / \partial t^2 = a^2 (\partial^2 u / \partial x^2)$, $x > 0, t > 0, u_x(0, t) = A \sin wt, u(x, 0) = 0, u_t(x, 0) = 0, |u(x, t)| < M$.
9. Solve the boundary value problem $\partial^2 u / \partial x^2 = (l/k) \times (\partial u / \partial t)$ with the conditions $u = 0$, when $x = \infty, u = 0$ when $t = 0$.
10. Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = a$ are kept at temperature zero and whose initial temperature is $\sin(\pi x/a)$.

11. A semi-infinite solid $x > 0$ is initially at temperature zero. At time $t = 0$, a constant temperature $u_0 = 0$ is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid at later time $t > 0$.
12. A bar of length of l is at constant temperature u_0 . At $t = 0$ the end $x = l$ is suddenly given the constant temperature u_l and the end $x = 0$ is insulated. Assuming that the surface of the bar is insulated. Find the temperature at any point x the bar at any time $t > 0$. Given that

$$L^{-1} \left\{ \frac{\cosh x \sqrt{s}}{\cosh a \sqrt{s}} \right\} = 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{(2n-1)^2 \pi^2 t / 4a^2} \cos \frac{(2n-1)\pi x}{2a}$$

13. A semi-infinite solid $x > 0$ has its initial temperature equal to zero. A constant heat flux A is applied at the face $x = 0$ so that $-K u_x(0, t) = A$. Find the temperature at any point

$$x > 0 \text{ of solid given that } L^{-1} \left\{ \frac{e^{-x\sqrt{s}}}{s^{3/2}} \right\} = \left(\frac{t}{\pi} \right)^{1/2} e^{-x^2/4t^2} - \frac{x}{2} \operatorname{erfc} \left(\frac{x}{2} \sqrt{t} \right)$$

14. The semi-infinite solid $x > 0$ has thermal diffusivity k_1 and conductivity K_1 . It is in contact along the plane $x = 0$ with the semi-infinite solid $x > 0$ in which these quantities are k_2, K_2 . If the initial temperature of the first solid is constant V_0 and that of the second solid is zero show that the temperature at time t in the second solid is
15. Solve the wave equation $\partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2), x > 0, t > 0$, where $u(x, 0) = 0, u_t(x, 0) = x > 0$ and $u(0, t) = F(t), \lim_{x \rightarrow \infty} u(x, t) = 0, t > 0$
16. A string is stretched between two fixed points $(0, 0)$ and $(a, 0)$. If it is displaced into the curve $u = b \sin(\pi x/a)$ and released from rest in that position at time $t = 0$, find its displacement at any time $t > 0$ and at any point $0 < x < a$.
17. Solve the boundary value problem: $\partial^2 u / \partial t^2 = a^2 (\partial^2 u / \partial x^2) - g, x > 0, t > 0$ with the boundary conditions $u(x, 0) = 0 = u_t(x, 0), x > 0; u(0, t) = 0, \lim_{x \rightarrow \infty} u(x, t) = 0, t \geq 0$.
18. An infinite long string having one end at $x = 0$ is initially at rest on the x -axis. The end $x = 0$ undergoes a periodic transverse displacement given by $A_0 \sin nt, t > 0$. Find the displacement of any point on the string at $t > 0$.
19. Solve the boundary value problems. $\partial^2 u / \partial t^2 = a^2 (\partial^2 u / \partial x^2) + g, t > 0, 0 < x < l$ with $u = \partial u / \partial t = 0$ when $t = 0, 0 < x < l; u = 0$, when $x = 0$ and $\partial u / \partial x = 0$ when $x = l$, for $t > 0$ given

$$L^{-1} \left\{ \frac{\cosh sx}{s^2 \cosh sl} \right\} = \frac{t^2 + x^2 + l^2}{2} - \frac{16 t^2}{\pi^3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos \frac{(2n-1)\pi x}{2l} \cos \frac{(2n-1)\pi t}{2l}$$

20. Solve $\partial u / \partial t = \partial^2 u / \partial x^2$, $x > 0, t > 0$, subject to condition

$$u(0, t) = 0, u(x, 0) = \begin{cases} l, & 0 < x < 1 \\ 0, & x > 1 \end{cases}, u(x, t) \text{ isd bounded}$$

21. Solve $\partial U / \partial t = 2(\partial^2 U / \partial x^2)$, if $U(0, t) = 0, U(x, 0) = e^x, x > 0, U(x, t)$ in bounded where $x > 0, t > 0$.

22. Using the Fourier sine transform, solve the partial differential equation $\partial V / \partial t = k(\partial^2 V / \partial x^2)$ for $x > 0, t > 0$, under the boundary conditions $V = V_0$ when $x = 0, t > 0$ and the initial condition $V = 0$, when $t = 0, x > 0$.

23. Show that the solution of Laplace equation for V inside the semi-infinite strip $x < 0, 0 < y < b$ such that $V = f(x)$, when $y = 0, 0 < x < \infty$; $V = 0$, when $y = b, 0 < x < \infty$;

$V = 0$, when $x = 0, 0 < y < b$, is given by.

$$V = \frac{2}{\pi} \int_0^{\infty} f(u) du \int_0^{\infty} \frac{\sin h(b-y)s}{\sin sb} \sin xs \sin us ds,$$

24. The temperature U in the semi-infinite rod $0 \leq x < \infty$ is determined by the equation $\partial U / \partial t = k(\partial^2 U / \partial x^2)$ subject to the conditions.

(i) $U = 0$ when $t = 0, x \geq 0$

(ii) $\partial U / \partial x = -\mu$ (a constant), when $x = 0$ and $t > 0$,

Making use of cosine transform, show that $U(x, t) = \frac{2\mu}{\pi} \int_0^{\infty} \frac{\cos sx}{s^2} (1 - e^{-ks^2 t}) ds$

25. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0, t > 0$, subject to conditions $u_x(0, t) = 0, u(x, t)$ is bounded and $u(x, 0) = \begin{cases} x, & 0 \leq x \leq l \\ 0, & x > l \end{cases}$

26. Use a cosine transform to show that the steady temperature in the semi-infinite solid $y > 0$ when the temperature on the surface $y = 0$ is kept at unity over the strip $|x| < a$ and at zero outside the strip, is $\frac{l}{\pi} \left[\tan^{-1} \left(\frac{a+x}{y} \right) + \tan^{-1} \left(\frac{a-x}{y} \right) \right]$

The result $\int_0^{\infty} e^{-sx} x^{-1} \sin rx dx = \tan^{-1} \frac{r}{s}, r > 0, s > 0$ may be assumed.

27. (a) If the flow of heat is linear so that the variation of θ (temperature) with z and y may be neglected and if it is assumed that no heat is generated in the medium, then solve the differential equation $\partial \theta / \partial t = k(\partial^2 \theta / \partial x^2)$ where $-\infty < x < \infty$ and $\theta = f(x)$ being a given function of x .

(b) If the function $U(x, y)$ is determined by equation $\partial U / \partial x = \partial^2 U / \partial y^2$ for $x \geq 0, -\infty < y < \infty$ and $U = f(y)$ when $x = 0$, that

$U(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(x) e^{-x^2 - x - isy} ds$, where $\bar{f}(s)$ is the Fourier transform of $f(y)$.

28. Use the complex form of the Fourier transform to show that

$$V = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(u) \exp \left\{ -\frac{(x-u)^2}{4t} \right\} du$$

Is the solution of the boundary value problem

$$\partial V / \partial t = \partial^2 V / \partial x^2, \quad -\infty < x < \infty, \quad t > 0; \quad V = f(x); \quad \text{when } t = 0.$$

29. Use the method of Fourier transform to determine the displacement $y(x, t)$ of an infinite string. Given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$. Show that solution can also be put in the form $y(x, t) = [f(0x + ct) + f(x - ct)]/2$

Answer Key

1. $y(x, t) = -x \times \left(\frac{t^3}{3!} \right) = -\left(\frac{xt^3}{6} \right)$ which is the required solution.

2. $y(x, t) = 6e^{-3x} L^{-1} \{1/(s+2)\} = 6e^{-3x} e^{-2t} = 6e^{-(3x+2t)}$

3. (a) $\therefore y(x, t) = L^{-1} \{(1/s) \times e^{-x\sqrt{s}}\} = \text{erfc}(x/2\sqrt{t})$.

(b) $y(x, t) = 20 \sin 2\pi x L^{-1} \left\{ \frac{x}{s^2 + (6\pi)^2} \right\} = 20 \sin 2\pi x \cos \pi t$.

4. Inverting, $y(x, t) = 20 \cos 3x e^{-27t} - 5 \cos 9x e^{-243t}$.

5. $u(x, t) = 3(t - 4x)^2 H(t - 4x) - t^2$

6. Since $L^{-1}\{1/s^2\} = t$, so by second shifting theorem (refer Art. 2.9 chapter 2) we get

$u(x, t) = (t - x/a)H(t - x/a)$, where $H(t - x/a)$ is Heaviside's unit step function

7. $u(x - t) = E \sin a(t - x/c) H(t - x/c)$

8. $u(x, t) = -(Aa/w) \times \{1 - \cos w(t - x/a)\} H(t - xa)$

9. $\frac{e_2 x}{\sqrt{k}} \frac{1}{2\sqrt{(\pi t^3)}} \left[1 - \frac{(x^2/4kt)}{1!} + \frac{(x^2/4kt)^2}{2!} - \dots \right] = \frac{c_2 x}{\sqrt{k}} \times \frac{1}{2\sqrt{(\pi t^3)}} e^{-(x^2/4kt)}$

10. $u(x, t) = \sin(\pi x/a) e^{-(\pi^2 kt)/a^2}$

11. $u(x, t) = u_0 \text{erfc}(x/\sqrt{k})$.

12.

$$u = u_1 + \frac{4(u_1 - u_0)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2 \pi^2 kt/4t^2} \cos \frac{(2n-1)\pi x}{2l}$$

13. $\therefore u(x, t) = \frac{A\sqrt{k}}{K} L^{-1} \left\{ \frac{e^{-x\sqrt{x/k}}}{s^{3/2}} \right\} = \frac{A\sqrt{k}}{K} \left[\left(\frac{t}{\pi} \right)^{1/2} e^{-x^2/4kt^2} - \frac{x}{2\sqrt{k}} \text{erfc} \left(\frac{x}{2\sqrt{kt}} \right) \right]$

14. $L^{-1}\{(1/s) \times e^{-c\sqrt{s}}\} = \text{erfc}\{c/2\sqrt{t}\}$

15. Where $H(t - x/c)$ is the Heaviside unit step function.

$$16. u(x, t) = b \sin \frac{\pi x}{a} L^{-1} \left\{ \frac{S}{S^2 + (\pi c/a)^2} \right\} b \sin \frac{\pi x}{a} \cos \frac{\pi c t}{a}.$$

$$17. u(x, t) = (1/2) \times g(t - x/a)^2 H(t - x/a) - (1/2) \times g t^2$$

$$18. u(x, t) = A_0 n \times \frac{1}{n} \sin n \left(t - \frac{x}{c} \right) H \left(t - \frac{x}{c} \right) \text{ as } L^{-1} \left\{ \frac{1}{S^2 + n^2} \right\} = \frac{\sin n t}{n}$$

or

$u(x, t) = A_0 \sin n (t - x/c) H(t - x/c)$, where $H(t - x/c)$ is the Heaviside unit step function.

19.

$$u(x, t) = -\frac{g}{2} \left\{ \left(\frac{l-x}{a} \right)^2 - \frac{l^2}{a^2} \right\} + \frac{16 g l^2}{a^2 \pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos \frac{(2n-1)(l-x)\pi}{2l} \cos \frac{(2n-1)\pi t a}{2l}$$

$$20. u(x, t) = \int_0^{\infty} \frac{1 - \cos s}{s} e^{-s^2 t} \sin s x \, dx, \text{ which is the required solution.}$$

$$21. \bar{U}_s(s, t) = \{S/(1 + S^2)\} e^{-2s^2 t}$$

$$22. V(x, t) = V_0 \left[1 - \frac{2}{\pi} \int_0^{\infty} \frac{1}{2} e^{-ks^2 t} \sin s x \, ds \right]$$

$$V(x, t) = V_0 \operatorname{erfc} (x/2\sqrt{kt})$$

$$23. V(x, y) = \frac{2}{\pi} \int_0^{\infty} f(u) du \int_0^{\infty} \frac{\sin h(b-t)s}{\sin h s b} \sin s u \sin s x \, ds, \text{ on changing the order of integration}$$

$$24. U(x, t) = \frac{2}{\pi} \int_0^{\infty} \frac{\mu}{s^2} (1 - e^{-ks^2 t}) \cos s x \, ds = \frac{2\mu}{\pi} \int_0^{\infty} (1 - e^{-ks^2 t}) ds$$

$$25. u(x, t) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2} \right) e^{-s^2 t} \cos s x \, dx$$

$$26. \int_0^{\infty} e^{-sx} x^{-1} \sin r x \, dx = \tan^{-1} \frac{r}{s}$$

$$27. (a) \bar{f}(s) = \int_{-\infty}^{\infty} f(x) e^{tsx} \, dx.$$

$$(b) V(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s) \cos s^2 y e^{-isx} \, ds, \text{ where } \bar{f}(s) \text{ is given by (10).}$$

$$28. = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(u) \exp \left\{ -\frac{(x-u)^2}{4t} \right\} du.$$

$$29. = [f(x + ct) + f(x - ct)]/2, \text{ by inversion formula for Fourier transform.}$$