COMPLEX MATRICES

If at least one element of a matrix is a complex number $a + ib$, where *a, b* are real andi $= \sqrt{-1}$, then the matrix is called a complex matrix.

The matrix obtained by replacing he elements of a complex matrix A by the corresponding conjugate complex number is called the conjugate of the matrix A and is denoted by $\bar{A}$.

Thus, if $A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 2-3i & 7i \\ 5 & 1+i \end{bmatrix}$

It is easy to see that the *conjugate of the transpose o*f $i.e., (\overline{A'})$ and the *transposed conjugate of* A $i.e., (\bar{A}')$ are equal, Each of them is denoted by A^*.

Thus, $(\bar{A}')\ (\bar{A}') = A^*$.

A square matrix $A = [a_{ij}]$is said to be Hermitian if $A *= A$ or $a_{ij} = a_{ji}$.

A square matrix $A = [a_{ij}]$ is said to be skew. Hermitian if $A^* = -A$ or $a_{ij} = -\bar{a}_{ji.}$

In a Hermitian matrix, the diagonal elements are all real, while every other element is the conjugate complex of the element in the transposed position. For example.

$A = \begin{bmatrix} 5 & 2+i & -3i \\ 2-i & -3 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ is a Hermitian matrix.

In a skew-Hermitian matrix, the diagonal elements are zero or purely imaginary of the form $i\beta$, where β is real. Every other element is the negative of the conjugate complex of the element in the transposed position.

For example, $B = \begin{bmatrix} 3i & 1+i & 7 \\ -1+i & 0 & -2-i \\ -7 & 2-i & -i \end{bmatrix}$ is a skew-Hermitian matrix

A square matrix A is said to be unitary if $AA^* = I = A^*A$

The determinant of a unitary matrix is of unit modulus. For a matrix to be unitary, it must be non-singular.

A square matrix A is said to be orthogonal if AA'=I= A'A or A' =A^{-1}

Note. The following results hold:

(i) $(\bar{A}) = A$ (ii) $\overline{A+B} - \bar{A} + \bar{B}$ (iii) $\overline{\lambda A} = \overline{\lambda A}$

(iv) $\overline{AB} = \overline{AB}$ (v) $(A^*)^*$=A (vi) $(A+B)^* = A^*+B^*$

(vii) $(\lambda A)^* = \bar{\lambda}A^*$ (viii) $(AB)^* = B^*A^*$

ILLUSTRATIVE EXAMPLES

Example1. *If* $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, *verify that* A^*A *is a Hermitian matrix where* A^* *is the conjugate transpose of A.*

Sol. $$A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$$

$$A^* = (\bar{A'}) = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

Now, $$A^*A = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix} \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix} = B(say)$$

$\therefore$ $$B' = \begin{bmatrix} 30 & 6+8i & -19-17i \\ 6-8i & 10 & -5-5i \\ -19+17i & -5+5i & 30 \end{bmatrix}$$

Now, $$B^* = (\overline{B'}) = \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix} = B$$

$\Rightarrow$ $B = A^*A$ is a Hermitian matrix.

Example 2. *Show that the matrix*

$$\begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix} \textit{is unitary if } \alpha^2+\beta^2+\gamma^2+\delta^2 = 1.$$

Sol. Let $$A = \begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}$$

$\therefore$ $$\bar{A} = \begin{bmatrix} \alpha-i\gamma & -\beta-i\delta \\ \beta-i\delta & \alpha+i\gamma \end{bmatrix}$$

$\Rightarrow$ $$A^* = (\bar{A})' = \begin{bmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{bmatrix}$$

For a square matrix A to be unitary,

$$AA^* = I = A^*A \qquad \ldots(1)$$

Now, $$AA^* = \begin{bmatrix} \alpha^2+\beta^2+\gamma^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{bmatrix}$$

Also, $$A^*A = \begin{bmatrix} \alpha^2+\beta^2+\gamma^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{bmatrix}$$

Eqn. (1) is satisfied only when

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = 1$$

Example 3. *If A and B are Hermitian, show that AB-BA is skew-Hermitian.*

Sol. A and B are Hermitian ⇒ $A^* = A$ and $B^* = B$

Now, $(AB - BA)^* - (BA)^*$

$$B^*A^* - A^*B^* = BA - AB = -(AB - BA)$$

⇒ AB - BA is skew-Hermitian.

Example 4. *If A is* α *skew-Hermitian matrix* ⇒ $A^* = -A$

Now, $$(iA)^* = \bar{\imath}\,A^* = (-i)(-A) = iA$$

⇒ iA is a Hermitian matrix.

Example 5. *Show that every square motrix is expressible as the sum of of a Hermitian matrix and a skew-0 Hermitian matrix.*

Sol. Let A be any square matrix.

Since $$(A + A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$$

and $$(A - A^*)^* = A^* - (A^*)^* = A^* - A = -(A - A^*)$$

∴ $A + A^*$ is Hermitian and $A - A^*$ is skew-Hermitian.

Now, $$A = \tfrac{1}{2}(A + A^*) + \tfrac{1}{2}(A - A^*) = P + Q \text{ (say)}$$

where P is Hermitian and Q is skew- Hermitian. Thus, every square matrix can be expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Example 6. *If* $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, *obtain the matrix* $(I - N)(I + N)^{-1}$, *and shw that it is unitary*

Sol. $$1 - N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$1 + N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I + N| = \begin{vmatrix} 1 & 1+2i \\ -1+2i & 1 \end{vmatrix} = 1 - (4i^2 - 1) = 6$$

$$(I + N)^{-1} \frac{1}{|I+N|} adj\,(I + N) = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$\therefore \qquad (I-N)(I+N)^{-1} = \begin{vmatrix} 1 & -1-4i \\ 1-2i & 1 \end{vmatrix} \frac{1}{6} \begin{vmatrix} 1 & -1-2i \\ 1-2i & 1 \end{vmatrix}$

$$= \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = A(say)$$

$$A' = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2-4i & -1 \end{bmatrix}$$

$$\overline{(A')} = A^{\neq} = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$A^{\neq}A \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

$$\frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow \qquad A = (I-N)(I+N)^{-1}$ is unitary

TEST YOUR KNOWLEDGE

1. Show that the matrix A is Hermitian and iA is Skew-Hermitian where A is.

(i) $\begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 5+2i & -3 \\ 5-2i & 7 & 4i \\ -3 & -4i & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$

2. (i) Express the matrix $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$. as a sum of Hermitian and Skew-Hermitian matrix.

(ii) Express the Hermitian matrix $A = 1\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$ as $P + iQ$ where P is a real symmetric and Q is a real skew-symmetric matrix.

3. Show that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is skew- Hermitian and also unitary

4. (i) If A is any square matrix, prove that $A + A^*, AA^*, A^*A$ are all Hermitian and $A - A^*$ is Skew-Hermitian.

(ii) If A, B are hermitian or Skew-Hermitian, then so is A + B.

(iii) Show that the matrix B^*AB is Hermitian or Skew-Hermitian as A is Hermitian or Skew-Hermitian.

(iv) If A is a Hrmitian matrix, then show that iA is a Skew-Hermitian matrix.

5. (i) Define unitary matrix. Show that the following matrix is unitary.

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

(ii) Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

(iii) Show that $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ is a unitary matrix, where ω is complex cube root of unity.

6. Verify that the matrix $A = \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}$ have eigen values with unit modulus.

1.36. CHARACTERISTIC EQUATION:

If A is square matrix of order n, we can form the matrix $A - \lambda I$, where λ is a scalar and I is the unit matrix of order n, The determinant of this matrix equated to zero, $i.e.$,

$|A - \lambda I| = \begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda \dots & & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix} = 0$ is called the characteristic equation of A.

On expanding the determinant, the characteristic equation can be written as a polynomial equation of degree n in λ of the form $(-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \cdots + k_n = 0$.

The roots of this equation are called the *characteristic roots or latent roots or eigen values* of A.

The set of eigen values of a square matrix A is called the spectrum of A.

Note. The sum of the eigen values of a matrix A is equal to trace of A.

[The trace of square matrix I the sum of the diagonal elements]

1.37. EIGEN VECTORS

Consider the linear transformation $Y = AX$ …(1)

Which transforms the column vector X into the column vector Y. In practice, we are often required to find those vectors X which transform into scalar multiples of themselves.

Let X be such a vector which transforms into λX (λ being a non -zero scalar) by the transformation (1).

Then $\quad Y = \lambda X \quad \ldots(2)$

From (1) and (2), $\quad AX = \lambda X \Rightarrow AX - \lambda IX = O \Rightarrow (A - \lambda I)X = O \quad \ldots(3)$

This matrix equation gives n homogeneous linear equations

$$\left.\begin{array}{llll} (a_{11}-\lambda)x_1 & +a_{12}x_2 & +\cdots + a_{1n}x_n = 0 \\ a_{21}x_1 & +(a_{22}-\lambda)x_2 & +..+a_{2n}x_n = 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{n1}x_1 & +a_{n2}x_2 & +..+(a_{nn}-\lambda)x_n = 0 \end{array}\right\} \quad \ldots(4)$$

These equations will have a not-trivial solution only if the co-efficient matrix $A - \lambda I$ is singular

$i.e.$ if $\quad |A - \lambda I| = 0 \quad \ldots(5)$

This is the characteristic equation of the matrix A and has n roots which are the eigen values of A. Corresponding to each root of (5), the homogeneous system (3) has a non-zero solution.

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ which is called an eigen vector or latent vector.

Note. If X is a solution of (3), then so is kX, where l is an arbitrary constant. Thus, the eigen vector corresponding to an eigen value is not unique.

1.38. THE CHARACTERISTIC ROOTS OF A UNITARY MATRIX ARE OF UNIT MODULUS

Let A be a unitary matrix so that

$$A^*A = I = AA^* \quad \ldots(1)$$

If λ is a characteristic root of the matrix A and X is its latent vector, then we have

$$AX = \lambda X \quad \ldots(2)$$

Taking transpose conjugate of (2), we obtain

$$(AX)^* = (\lambda x)^* \quad \ldots(3) \; |\therefore \lambda^* = \lambda^-$$

$$X^*A^* = \bar{\lambda}X^*$$

On multiplying (2) and(3), we get

$$(X^*A^*)(AX) = (\bar{\lambda}X^*)(\lambda X)$$

$$\Rightarrow \quad X^*(A^*A)X = \lambda\bar{\lambda}\,(X^*X) \quad |Using\ (1)$$

$$\Rightarrow \quad X^*X = \lambda\bar{\lambda}(X^*X) \quad \ldots(4)$$

$$\Rightarrow \quad (1 - \lambda\bar{\lambda})X^*X = 0$$

Since X is a characteristic vector, $X \neq O$

Consequently, $X^*X \neq O$

Hence equation (4) gives

$$1 - \lambda\bar{\lambda} = 0$$

$\Rightarrow$ $\lambda\bar{\lambda} = 1$

$\Rightarrow$ $|\lambda|^2 = 1 \quad \Rightarrow |\lambda| = 1$

Hence the characteristic roots of a unitary matrix are of unit modulus

1.39. THE LATENT ROOTS OF A HERAMITIAN MATRIX ARE ALL REAL

Let λ be the characteristic or latent root of a Hermitian matrix A. Then $\exists$ a non-zero latent vector X such that

$$AX = \lambda X \qquad \ldots(1)$$

Pre-multiplying both sides of (1) by X^*, we get

$$X^*AX = X^*\lambda X \qquad \ldots(2)$$

Transpose conjugate of (2) gives

$$(X^*AX)^* = (\lambda X^*)^*$$

$\Rightarrow$ $X^*A^*(X^*)^* = X^*(X^*)^*\lambda^*$ $\quad |By\ reversal\ law$

$\Rightarrow$ $X^*A^*X = X^*X\bar{\lambda}$ $\quad |\because \lambda^* = \bar{\lambda}$

But A is a Hermitian matrix so that A^*A

Hence above equation becomes

$$X^*AX = \bar{\lambda}X^*X \qquad \ldots(3)$$

From (2) and (3), we have

$$X^*(\bar{\lambda}X^*X)$$

$\Rightarrow$ $(\lambda - \bar{\lambda})X^*X = 0$ $\qquad \ldots(4)$

Since X is a non-zero latent vector

$\therefore$ $X^*X \neq O$

Hence from (4), we have

$$\lambda - \bar{\lambda} = 0 \qquad \Rightarrow \lambda = \bar{\lambda}$$

Which is possible only when λ is real.

Hence the latent roots of a Hermitian matrix are all real.

1.40. THE CHARACTERISTIC ROOTS OF A SKEW-HERMITIAN MATRIX IS EITHER ZERO OR PURELY AN IMAGINARY NUMBER

Since A is a Skew- Hermitian matrix

$\therefore$ iA is Hermitian matrix.

Let λ be a characteristic root of A.

Then, $AX = \lambda X \quad \Rightarrow \; (iA)x = (i\lambda)X$

$\Rightarrow i\lambda$ is a characteristic root of matrix iA.

But $i\lambda$ is a Hermitian matrix.

Therefore, $i\lambda$ should be real.

Hence, λ is either zero or purely imaginary.

1.41. THE CHARACTERISDTIC ROOTS OF AN IDEMPOTERNT MATRIX ARE EITHER ZERO OR UNITY

Since A is an idempotent matrix. $\therefore$ $A^2 = A$.

Let X be a latent vector of the matrix A corresponding to the latent root λ so that

$$AX = \lambda X \quad \ldots(1)$$

$\Rightarrow$ $(A - \lambda I)X = O$ such that $X \neq O$

Per - multiplying (1) by A

$$A(AX) = A(\lambda X) = \lambda(AX)$$

$\Rightarrow$ $(AA)X = \lambda(\lambda X)$ $\quad |by\ (1)$

$\Rightarrow$ $A^2X = \lambda^2 X \quad \Rightarrow AX = \lambda^2 X$ $\quad |\because A^2 = A$

$\Rightarrow$ $\lambda X = \lambda^2 X$ $\quad |by\ (1)$

$\Rightarrow$ $(\lambda^2 - \lambda)X = O \quad \Rightarrow \lambda^2 - \lambda = 0$ $\quad (Since\ X \neq O)$

$\Rightarrow$ $\lambda(\lambda - 1) = 0$

$\Rightarrow$ $\lambda = 0, 1$

ILLUSTRATIVE EXAMPLES

Example 1. *Show that if $\lambda_p \lambda_{2,} \ldots \lambda_n$ are the latent roots of the matrix A, then A^3 has the latent roots $\lambda_1^3, \ \lambda_2^3, \ \ldots \ \lambda_n^3$.*

Sol. Let λ be a latent root of the matrix A. Then there exists a non- zero vector X such that

$$AX = \lambda X \qquad \ldots(1)$$

$\Rightarrow \qquad A^2(AX) = A^2(\lambda X) \qquad \Rightarrow A^3X = \lambda(A^2X)$

But $\qquad A^2X = A(AX) = A(\lambda X) \qquad |Using\ (1)|$

$$= \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$$

$\therefore \qquad A^2X = \lambda(\lambda^2 X) = \lambda^3 X$

$\Rightarrow \qquad \lambda^3$ is a latent root of A^3.

$\therefore \qquad$ If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the latent roots of A, then $\lambda_1^3, \lambda_2^3, \ldots \ \lambda_n^3$ are then latent roots of A^3.

Example2. *If $\lambda_1, \lambda_2 \ldots \lambda_n$ are eigen values of A, then find eigen values of the matrix $(A - \lambda I)^2$*

Sol. $\qquad (A - \lambda I)^2 = A^2 - 2\lambda AI + \lambda^2 I^2$

$$= A^2 - 2\lambda A + \lambda^2 I$$

Eigen values of A^2 are $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$

Eigen values of $2\lambda A$ are $2\lambda\lambda_1, 2\lambda\lambda_2, \ldots \lambda\lambda_n$.

Eigen values of $\lambda^2 I$ are λ^2

$\therefore \qquad$ Eigen values of $(A - \lambda I)^2$ are

$$\lambda_1^2 - 2\lambda\lambda_1 + \lambda^2, \lambda_2^2 - 2\lambda\lambda_2 + \lambda^2, \ldots, \lambda_n^2 - 2\lambda\lambda_n + \lambda^2$$

or $\qquad (\lambda_1 - \lambda)^2, (\lambda_2 - \lambda)^2, \ldots, (\lambda_n - \lambda)^2$.

Example3. *If λ is an eigen value of a non-singular matrix A, show that*

(i) λ^{-1} is an eigen value of A^{-1}.

(ii) $\frac{|A|}{\lambda}$ is an eigen value of adj. A.

Sol. (i) λ is an eigen value of A

$\Rightarrow$ There exists a non-zero matrix X such that AX =λX

$\Rightarrow \qquad X = A^{-1}(\lambda X)$

$\Rightarrow \qquad X = \lambda(A^{-1}X)$

$\Rightarrow \qquad \frac{1}{\lambda}X = A^{-1}X$

$\Rightarrow \qquad A^{-1}X = \lambda^{-1}X$

$\Rightarrow$ λ^{-1} is an eigen value of A^{-1}.

(ii) λ is an eigen value of A

$\Rightarrow$ There exists a non-zero matrix X such that AX = λX

$\Rightarrow \qquad (adj.A)(AX) = (adj.A)(\lambda X)$

$\Rightarrow \qquad \{(adj.A)\}X = \lambda(adj.A)X$

$\Rightarrow \qquad |A|IX = \lambda(adj.A)X \qquad [\because \ (adj.A)A = |A|\, I]$

$\Rightarrow \qquad |A|X = \lambda(adj.A)X$

$\Rightarrow \qquad \frac{|A|}{\lambda}X = (adj.A)X$

$\Rightarrow \qquad (adj.A)X = \frac{|A|}{\lambda}X$

$\Rightarrow$ $\frac{|A|}{\lambda}$ is an eigen value of adj. A.

Example4. *Find the eigen values of the following matrices:*

(i) $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 5 & 7 \\ 5 & 3 & 1 \\ 7 & 0 & 2 \end{bmatrix}$

Sol. (i) Characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 9 & 4 \\ 0 & 2-\lambda & 0 \\ 3 & 1 & -3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \lambda^3 - 19\lambda + 30 = 0$$

$$\Rightarrow \lambda = 3, -5, 2$$

(ii) Characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 5 & 7 \\ 5 & 3-\lambda & 1 \\ 7 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 - 58\lambda + 150 = 0$$

$$\Rightarrow \lambda = 11.026, -6.215, 2.1888$$

Example5. *Show that for any square matrix A,*

(i) $A\ adn\ A'jave\ same\ set\ of\ eigen\ values.$

(ii) *The product of all eigen values of A is equal to determinant (A).*

Sol. Let A be a square matrix.

(i) The characteristic equation of A is $|A - \lambda I| = 0$ …(1)

Let A' be the transpose of A.

Then the characteristic equation of A' will be

$$|A' - \lambda I| = 0 \qquad \ldots(2)$$

Since the interchange of rows and columns does not alter the value of the determinant we have,

$$|A' - \lambda I| = |A - \lambda I|$$

$$|\because |A - \lambda I| = |A - \lambda I|' = |A' - \lambda I'| = |A' - \lambda I| \text{ as } I' = I$$

Hence the eigen values of matrix A and its transpose A' are identical.

(ii) Let $A = [a_{ij}]_{n\times n}$ be a given square matrix and $\lambda_1, \lambda_2, \lambda_{3,\ldots}\lambda_n$ be its eigen values. If $\phi(\lambda)$ be the characteristic polynomial then,

$$\phi(\lambda) = |A - \lambda I|$$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots. & a_{1n} \\ a_{12} & a_{22} - \lambda & \ldots. & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots. & a_{nn} - \lambda \end{vmatrix}$$

$$= (-1)^n\{\lambda^n + P_1\lambda^{n-2} + P_2\lambda^{n-2} + \cdots + P_n\}$$

$$= (-1)^n\{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\ldots(\lambda - \lambda_n)\}$$

Putting $\lambda = 0$, we get

$$\phi(0) = (-1)^n(-1)^n\lambda_1\lambda_2\lambda_3\ldots\lambda_n$$

$$|A| = \lambda_1\lambda_2\lambda_3\ldots\lambda_n$$

Hence the product of all eigen values of A is equal to determinant (A).

Example6.*Show that for a square matrix,*

(i) *There are infinitely many eigem vectors corresponding to a single eigen value*

(ii) *Every eigen vector corresponds to a unique eigen value.*

Sol. (i) Let X be a characteristic vector of a square matrix A correspoinding to a single eigen value λ. Then we have,

$$AX = \lambda X$$

Let k be any non -zero scalar. Than,

$$k(AX) = k(\lambda X) \qquad \Rightarrow A(kX) = \lambda(kX)$$

Therefore, kX is also a characteristic vector of A corresponding to the same characteristic root λ

Since k is any non-zero scalar, ∃ infinitely many eigen vectors corresponding to a single eigen value

(ii) Let there exist two distinct eigen values λ_1 and λ_2 corresponding to an eigen vector X of a square matrix A. Then, we have

$$AX = \lambda_1 X \qquad |\lambda_1 \neq \lambda_2$$

$$AX = \lambda_2 X$$

$\therefore$ $$AX = \lambda_1 X = \lambda_2 X$$

$\Rightarrow$ $$(\lambda_1 - \lambda_2)X = O$$

$\Rightarrow$ $$X = O \qquad |\because \ \lambda_1 - \lambda_2 \neq 0$$

Which is impossible since X is a non-zero vector. Hence every eigen vector corresponding to a unique eighen value

Example 8. *Find the eigen values and eigen vectors of matrix* $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

Sol. The characteristic equation of the given matrix is

$$|A = \lambda I| = 0$$

Or $$\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$\lambda^2 - 5\lambda - 6 = 0$$

$\Rightarrow$ $$\lambda = 6, -1.$$

Thus, the eigen values of A are $6, -1$,

Corresponding to $\lambda = 6$, the eigen vectors are given by

$$(A - 6I)X_1 = O$$

or $$\begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O$$

or $$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O$$

We get only one independent equation $-5x_1 - 2x_2 = 0$

$\Rightarrow$ $$\frac{x_1}{2} = \frac{x_2}{-5} = k_1(say)$$

$$x_1 = 2k_1, \qquad x_2 = -5k_1$$

$\therefore$ The eigen vectors are $X_2 = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example 8. *Find the eigen values and eigen vectors of the matrix is.*

Sol. The characteristic equation of the given matrix is

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

or $$(-2-\lambda)[-\lambda(1-\lambda) - 12] - 2[-2\lambda - 6] - 3[-4 + 1(1-\lambda)] = 0$$

or $$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

By trail, $\lambda = -3$ satisfies it.

$\therefore$ $(\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0 \Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0 \Rightarrow \lambda = -3, -3, 5$

Thus, the eigen values of A are $-3, -3, 5$.

Corresponding to $\lambda = -3$, the eigen vectors are given by

$$(A + 3I)X_1 = O$$

or $$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

We get only one independent equation $x_1 + 2x_2 - 3x_3 = 0$

Let $x_3 = k_2, \quad x_2 = k_2$ then $x_1 = 3k_1 - 2k_2$

$\therefore$ The eigen vectors are given by

$$X_1 = \begin{bmatrix} 3k_1 - 2k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Corresponding to $\lambda = 5$, the eigen vectors are given by $(A - 5I)X_2 = O$

$\Rightarrow$ $$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 = \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$-7x_1 + 2x_2 - 3x_3 = 0$$

$$x_1 - 2x_2 - 3x_3 = 0$$

$$x_1 - 2x_2 - 5x_3 = 0$$

From first two equation,

$$\frac{x_1}{10-6} = \frac{x_2}{3+5} = \frac{x_3}{-2-2}$$

$\Rightarrow$ $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k_3$ (say)

$\therefore$ $x_1 = k_3, x_2 + 2k_3, x_3 = -k_3$

Hence the eigen vectors are given by.

$$X_2 = k_3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Example. 9. *Prove that*

(i) *0 is a characteristic root of a matrix if and only if the matrix is singular.*

(ii) *The characteristic root of a real symmetric matrix are all real.*

(iii) *If A and B are square matrices of same type and if P be invertible then A and P^{-1} AP have same eigen values.*

(iv) *The sum of the eigen values of a square matrix is equal to the sum of the elements of its principal diagonal.*

Sol. (i) The characteristic roof of a matrix A is given by $|A - \lambda I| = 0$

If $\lambda = 0$, then it gives $|A| = 0$

$\Rightarrow$ A is singular.

Again if matrix A is singular, then

$$|A - \lambda I| = 0$$

$\Rightarrow$ $|A| - \lambda|I| = 0$

$\Rightarrow$ $0 - \lambda . I = 0$

$\Rightarrow$ $\lambda = 0$

(ii) Let A be a real symmetric matrix

$\because$ A is real

$\therefore$ $\overline{A} = A$

$\Rightarrow$ $\overline{(A)'} = A'$

$\Rightarrow$ $A^* = A$ $\quad |\because A \text{ is symmetric}$

$\Rightarrow$ A is Hermitian.

Hence the characteristic roots of A are all real

(iii) Let $$B = P^{-1}AP$$

Then $$B - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P$$

$\therefore$ $$|B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}|\,|A - \lambda I||P|$$

$$= |A - \lambda I||P^{-1}|\,|P| = |A - \lambda I||P^{-1} - P|$$

$$= |A - \lambda I|\,|I| = |A - \lambda I| \qquad |\because |I| = 1$$

Hence matrices A and $P^{-1}AP$ have the same characteristic roots.

(iv) Le$A = [a_{ij}]_{3\times3}$ be a square matrix of order 3. The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \cdots = 0 \qquad \ldots (1)$$

But we know that,

$$|A - \lambda I| = (-1)^3(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) \ldots. \qquad \ldots(2)$$

Comparing equations (1)and (2), we get

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$$

TEST YOUR KNOWLEDGE

1. Fine the eigen values and corresponding eigen vectors of the following matrices:

(i) $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

2. (i) Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

(ii) Find the eigen values of the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

(iii) Find the eigen vectors for the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

3. Find the eigen values of $3A^3 + 5A^2 - 6A + 2I$ where $a = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

4. Find the eigen value of matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$.

5. (i) Show that if λ ios a characteristic root of the matrix A then $\lambda + k$ is a characteristic root of the matrix $A + kI$.

(ii) Show that if $\lambda_1 (1 \le i \le n)$ are the eigen values of a square matrix A then A^m gas the eigen values $\lambda_i^m (1 \le) i \le n$, m beigen a positive integer

(iii) Prove that the characteristic roots of a diagonal matrix are the diagonal elements of the matrix

6. Verify that the matrices $X = \begin{bmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & f & h \\ f & 0 & g \\ h & g & 0 \end{bmatrix}, Z + \begin{bmatrix} 0 & g & f \\ g & 0 & h \\ f & h & 0 \end{bmatrix}$ have same characteristic equation.

7. If $a + b + c = 0$, Find the characteristic roots of the matrix $A = \begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix}$

8. Prove that for matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$, all its eigen values are distinct and real. Hence find the corresponding eigen vectors.

9. Show that the matrix $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ has leas than three linearly independent eigen vectors. Also find them.

10. If $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $P = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$, Show that the transform of P byM $i.e.$, MPM^{-1} is a diagonal matrix and hence find the eigen values of P.

11. Find characteristic equation and eigen values of the matrix $A = \begin{bmatrix} 3 & 2 & 2 & -4 \\ 2 & 3 & 2 & -1 \\ 1 & 1 & 2 & -1 \\ 2 & 2 & 2 & -1 \end{bmatrix}$

12. If $A\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ and $B = 1 - \frac{1}{4}\ A$, then show that $\mu_i = 1 - \frac{1}{4}\lambda_i$, Where λ_i and μ_i are the eigen values of A and B respectively.

ANSWERS

1. (i) $-1, -6;\ k_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}, k_2\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (ii) $1, 6;\ k_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}, k_2\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

   (iii) $1, 2, 2\ ;\ k_1\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, k_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ (iv) $a, b, c\ ;\ k_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, k_2\begin{bmatrix} h \\ b-a \\ 0 \end{bmatrix}, k_3\begin{bmatrix} g \\ 0 \\ c-a \end{bmatrix}$

2. (i) $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 4$ (ii) $0, 1, -2$ (iii) $k_1\begin{bmatrix} 1 \\ 1-i \end{bmatrix}, k_2\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

3. $4, 110, 10$

4. 6

7. $\lambda = 0, = \left[\frac{3}{2}\ (a^2 + b^2 + c^2)\right]^{1/2}$

8. $3, -1, 1;\ k_1\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, k_2\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, k_3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

9. $2, 2, 3;\ k_1\begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}, k_2\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

10. a, b, c

11. $\lambda^4 - 7\lambda^3 + 17\lambda^2 - 17\lambda + 6 = 0; \lambda = 1, 1, 2, 3$