Chapter
2

Finite Differences

Consider the function y = f(x), where x is known as argument and y is called entry. Here, the values of the argument are at equal intervals

a, a+h, a+2h, a+3h .... *a+nh,*

and the corresponding values of y are

f(a), f(a+h), f(a+2h), f(a+3h) .... f(a+ nh)

then the following difference are called finite differences

f(a+ h) - f(a)

f(a+2h) -f(a+h)

f(a+3h) - f(a+2h)

....

....

f(a+ nh) - f(a+(n-1) h)

Forward Difference:

If the above differences are denoted by the forward difference operator Δ then these differences are known as forward differences

$$\Delta f(a) = f(a+h) - f(a)$$

$$\Delta f(a+h) = f(a+2h) - f(a+h)$$

$$\Delta f(a+2h) = f(a+3h) - f(a+2h)$$

....

....

$$\Delta f(a+\overline{n-1}h) = f(a+nh) - f(a+\overline{n-1}h)$$

In general $\Delta f(x) = f(x+h) - f(x)$

where Δ is an operator and is called a forward difference operator and h is known as the interval of differences.

First forward difference $\Delta f(a) = f(a+h) - f(a)$ .... (1)

Second forward difference $\Delta^2 f(a) = \Delta[\Delta f(a)]$ .... (2)

$$= \Delta[f(a+h) - f(a)]$$
$$= \Delta f(a+h) - \Delta f(a)$$
$$= [f(a+2h) - f(a+h)] - [f(a+h) - f(a)]$$
$$= f(a+2h) - 2f(a+h) + f(a) \quad \text{....................(2)}$$

Third Forward difference:

$$\Delta^3 f(a) = \Delta[\Delta^2 f(a)] \quad \text{......................... (3)}$$

Putting the value of $\Delta^2 f(a)$ from (2) in (3), we get

$$\Delta^3 f(a) = \Delta[f(a+2h) - 2f(a+h) + f(a)]$$

$$= \Delta f(a+2h) - 2\Delta f(a+h) + \Delta f(a)$$
$$= [f(a+3h) - f(a+2h)] - 2[f(a+2h) - f(a+h)] + f(a+h) - f(a)$$
$$= f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)$$

Similarly $\Delta^n f(a) = \Delta^{n-1}\Delta f(a) = \Delta^{n-1}[f(a+h) - f(a)]$

$$\Delta^n f(a) = \Delta^{n-1} f(a+h) - \Delta^{n-1} f(a)$$

Remark:

1) $\Delta f(a)$ mean $f(a)$ is to be subtracted from next entry.

2) The difference *f (a +h) - f(a)* is denoted by placing Δ before the second entry.

3) Δ^2 is not the square of the operator Δ but Δ^2 means Δ operated by Δ

Table for forward difference:

Argument x	*Entry f(x)*	*First forward difference* $\Delta f(x)$	*2nd forward Difference* $\Delta^2 f(x)$	*3rd forward difference* $\Delta^3 f(x)$	*4th forward difference* $\Delta^4 f(x)$
a	$f(a)$	$f(a+h)-f(a) = \Delta f(a)$	$\Delta f(a+h)-\Delta f(a) = \Delta^2 f(a)$	$\Delta^2 f(a+h) - \Delta^2 f(a) = \Delta^3 f(a)$	$\Delta^3 f(a+h) - \Delta^3 f(a) = \Delta^4 f(a)$
$a+h$	$f(a+h)$	$f(a+2h) - f(a+h) = \Delta f(a+h)$	$\Delta f(a+2h) - \Delta f(a+h) = \Delta^2 f(a+h)$	$\Delta^2 f(a+2h) - \Delta^2 f(a+h) = \Delta^3 f(a+h)$	
$a+2h$	$f(a+2h)$	$f(a+3h) - f(a+2h) = \Delta f(a+2h)$	$\Delta f(a+3h) - \Delta f(a+2h) = \Delta^2 f(a+2h)$		
$a+3h$	$f(a+3h)$	$f(a+4h) - f(a+3h) = \Delta f(a+3h)$			
$a+4h$	$f(a+4h)$				

Problem: Construct a forward difference table and find $\Delta^4 f(1)$ if $f(1) = 1, f(2) = 3, f(3) = 8, f(4) = 15, f(5) = 25$

Solution:

x	**f(x)**	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	1				
2	3	2			
3	8	5	3		
4	15	7	2	-1	
5	25	10	3	1	2

From the table we have $\Delta^4 f(1) = 2$.

Backward difference operator:

If the difference *f(x) - f(x-h)* is denoted by the backward difference operator ∇ then the difference $\nabla f(x) = f(x) - f(x-h)$ is called backward difference.

The operator ∇ is called as backward difference operator.
It is to be noted that it is only the notation which changes and not the difference $y_1 - y_0 = \Delta y_0 = \nabla y_1$

2nd Backward Difference:

$$\nabla^2 f(x) = \nabla[f(x)] = \nabla[f(x) - f(x-h)]$$
$$= \nabla f(x) - \nabla f(x-h)$$
$$= [f(x) - f(x-h)] - [f(x-h) - f(x-2h)]$$
$$= f(x) - 2f(x-h) + f(x-2h)$$

3rd Backward Difference:

$$\nabla^3 f(x) = \nabla^2[\nabla f(x)] = \nabla^2[f(x) - f(x-h)]$$

$$= \nabla^2 f(x) - \nabla^2 f(x-h)$$
$$= f(x) - 2f(x-h)] + f(x-2h) - [f(x-h) - 2f(x-h) + f(x-3h)]$$
$$= f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h)$$

In general,

$$\nabla^n f(x) = \nabla^{n-1}[\nabla f(x)] = \nabla^{n-1}[f(x) - f(x-h)]$$
$$= \nabla^{n-1} f(x) - \nabla^{n-1} f(x-h)$$

Remark: The backward difference *f(a) - f(a-h)* is denoted by placing backward difference operator ∇ before the Ist entry.

Table for backward difference.

Argument ***x***	***Entry f(x)***	***First forward difference*** $\nabla f(x)$	***2nd forward Difference*** $\nabla^2 f(x)$	***3rd forward difference*** $\nabla^3 f(x)$	***4th forward difference*** $\nabla^4 f(x)$
a	*f(a)*	$f(a+h)-f(a)$ $=$			
a+h	*f(a+h)*	$\nabla f(a+h)$	$\nabla f(a+2h)-\nabla f(a+$		
a+2h	*f(a+2h)*	$f(a+2h)$ $-f(a+h)$ $=$ $\nabla f(a+2h)$	$= \nabla^2 f(a+2h)$	$\nabla^2 f(a+3h)$ $-\nabla^2 f(a+2h)$ $= \nabla^3 f(a+3h)$	
a+3h	*f(a+3h)*	$f(a+3h)$ $-f(a+2h)$ $= \nabla f(a+3h)$	$\nabla f(a+3h)$ $-\nabla f(a+2h)$ $= \nabla^2 f(a+3h)$	$\nabla^2 f(a+4h)$ $-\nabla^2 f(a+3h)$ $= \nabla^3 f(a+4h)$	$\nabla^3 f(a+4h)$ $-\nabla^3 f(a+3h)$ $=$ $\nabla^4 f(a+4h)$
a+4h	*f(a+4h)*	$f(a+4h)$ $-f(a+3h)$ $= \nabla f(a+4h)$	$\nabla f(a+4h)$ $-\nabla f(a+3h)$ $= \nabla^2 f(a+4h)$		

Problem: Construct a backward difference table for
f (1) =4, f (2) =8, f (3) =2, f (4) = 18, f (5) = 36 find $\nabla^5 f(5)$.

Solution:

x	**f(x)**	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	4				
2	8	4			
3	12	4	0		
4	18	6	2	-2	
5	36	18	12	10	12

From the table, we have $\nabla^5 f(5) = 12$.

Shifting operator E:

$Ef(x) = f(x+h)$ .... (1)

The operator E is called the shifting operator E.

$\Delta f(x) = f(x+h) - f(x) \Rightarrow \Delta f(x) = Ef(x) - f(x)$ By (1)

$\Rightarrow f(x) = \Delta f(x) + f(x) \Rightarrow f(x) = (0+1)f(x)$

$\Rightarrow E = \Delta + 1$ or Δ=E-1

Again, $E^2 f(x) = EEf(x) = Ef(x+h) = f(x+2h)$

$E^3 f(x) = EE^2 f(x) = Ef(x+2h) = f(x+3h)$

$E^4 f(x) = EE^3 f(x) = Ef(x+3h) = f(x+4h)$

....

....

$E^n f(x) = f(x+nh)$

Similarly, we define $E^{-1} f(x) = \Delta f(x-h)$

In general $E^{-n} f(x) = f(x-nh)$

Properties of Δ and E.

(1) Δ and E are commutative with regard to constant

(1) $\Delta[af(x)] = a\Delta f(x)$

(2) $E[af(x)] = aEf(x)$

Proof: (1) $\Delta[af(x)] = af(x+h) - af(x)$

$= a[f(x+h) - f(x)] = a\Delta f(x)$

(2) $E[af(x)] = af(x+h) = aEf(x)$

(2) Commutative property of Δ and E.

$E\Delta f(x) = \Delta Ef(x)$

Proof: $E\Delta f(x) = E[f(x+h) - f(x)] = Ef(x+h) - Ef(x)$

$= f(x+2h) - f(x+h)$...............(1)

$\Delta E f(x) = \Delta f(x+h) = f(x+2h) - f(x+h)$...............(2)

From (1) (2) $E\Delta f(x) = \Delta Ef(x)$.

(3) Associative property.

Δ and E are associative

$$(\Delta E)\Delta f(x) = \Delta(E\Delta)f(x)$$

(4) Distributive property: Δ and E are distributive

(1) $\Delta[f(x)+\phi(x)] = \Delta f(x) + \Delta\phi(x)$

(2) $E[f(x)+\phi(x)] = Ef(x) + E\phi(x)$

Proof:

$$\Delta[f(x)+\phi(x)] = [f(x+h)+\phi(x+h)] - [f(x)+\phi(x)]$$
$$= [f(x+h)-f(x)] + [\phi(x+h)-\phi(x)]$$
$$= \Delta f(x) + \Delta\phi(x)$$

$$E[f(x)+\phi(x)] = f(x+h)+\phi(x+h)] = E\, f(x) + E\phi(x)$$

(5) Law of indices

(1) $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$

(2) $E^m E^n f(x) = E^{m+n} f(x)$

(6) Let k be any constant then

(1) $\Delta k = 0$

(2) $Ek = k$

Proof: Let $k = f(x)$, $k = f(x+h)$

(1) $\Delta k = \Delta f(x) = f(x+h) - f(x) = k - k = 0$

(2) $E(k) = Ef(x) = f(x+h) = k$

Central Difference operator: Central difference operator is defined as

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\delta f(x) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x) \qquad \left[\because E^n f(x) = f(x+nh)\right]$$

$$\delta f(x) = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x)$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

Central Difference Table:

Argument x	***Entry y***	***First central difference*** δy	***2nd central Difference*** $\delta^2 y$	***3rd central difference*** $\delta^3 y$	***4th central difference*** $\delta^4 y$
a	y_0				
a+h	y_1	$\delta y_{1/2}$	$\delta^2 y_1$		
a+2h	y_2	$\delta y_{3/2}$	$\delta^2 y_2$	$\delta^3 y_{3/2}$	$\delta^4 y_2$
a+3h	y_3	$\delta y_{5/2}$	$\delta^2 y_3$	$\delta^3 y_{5/2}$	
a+4h	y_4	$\delta y_{7/2}$			

Problem: Evaluate $\delta^4 f(2)$, given *f (0) = 8, f (1) = 12, f (2) = 20, f (3) = 34, f (4) = 60.*

Solution:

Argument	***Entry***	***First difference***	***2nd Difference***	***3rd difference***	***4th difference***
0	*8*				
1	*12*	$\delta y_{1/2} = 4$	$\delta^2 y_1 = 4$	$\delta^3 y_{3/2} = 2$	
2	*20*	$\delta y_{3/2} = 8$	$\delta^2 y_2 = 6$	$\delta^3 y_{5/2} = 6$	$\delta^4 y_2 = 4$
3	*34*	$\delta y_{5/2} = 14$	$\delta^2 y_3 = 12$		
4	*60*	$\delta y_{7/2} = 26$			

Thus: $\delta^4 f(2) = 4$.

Averaging operator: The averaging operator is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$$

Relation between different operators:

Relation between averaging operator and central difference operator δ **:**

$$\mu^2 = 1 + \frac{\delta^2}{4}$$

Proof: $\mu f(x) = \frac{1}{2}\left[E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)\right] = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right] f(x)$

$$\mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]$$

Now $\mu^2 f(x) = \frac{1}{4}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]^2 f(x)$

$$= \frac{1}{4}\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 + 4\right] f(x) = \frac{1}{4}\left[(\delta)^2 + 4\right] f(x)$$

$$\Rightarrow \mu^2 = 1 + \frac{\delta^2}{4}$$

Relation between E, δ *and* μ

We know that $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$

$$\Rightarrow \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} \quad \text{..............(1)}$$

And $\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$

$$\mu f(x) = \frac{1}{2}\left(E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)\right)$$

$$2\mu = E^{\frac{1}{2}} + E^{-\frac{1}{2}} \quad \text{..............(2)}$$

Adding (1) & (2) $\delta + 2\mu = 2E^{\frac{1}{2}}$

Or $E^{\frac{1}{2}} = \mu + \frac{\delta}{2}$

Relation between ∇ and E^{-1}

(1) $\nabla = 1 - E^{-1}$

(2) $E\nabla = \nabla E = \Delta$

Solution: (1) $\nabla f(a) = f(a) - f(a-h)$ .... (I)

& $E^{-1} f(a) = f(a-h)$ .... (II)

Putting the value of *f (a-h)* from (II) in (I), we get

$$\Delta f(a) = f(a) - E^{-1} f(a) = (1 - E^{-1}) f(a)$$

$$\Rightarrow \nabla = 1 - E^{-1}$$

$$\Rightarrow E^{-1} = 1 - \nabla$$

(2) $\nabla f(a) = E[f(a) - f(a-h)] = Ef(a) - E\delta(a-h)$

$$= Ef(a) - f(a) = (E-1) f(a) = \Delta f(a)$$

so $E\nabla = \Delta$............................(1)

$\nabla E f(a) = \nabla f(a+h) = f(a+h) - f(a) = \Delta f(a)$

so $\nabla E = \Delta$............................(2)

so from (1) and (2) $E\nabla = \nabla E = \Delta$

Relation between Δ and δ.

$$\Delta = \frac{\delta^2}{2} \pm \delta\sqrt{1 + \frac{\delta^2}{4}}$$

Solution: $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$

$$\Rightarrow \delta f(x) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

Squaring we get $\delta^2 = E + E^{-1} - 2$

$$E\delta^2 = E^2 + 1 - 2E \;\; = (E-1)^2$$

$$(1+\Delta)\delta^2 = \Delta^2$$

$$\Delta^2 - \delta^2\Delta - \delta^2 = 0$$

$$\Delta = \frac{\delta^2 \pm \sqrt{\delta^4 + 4\delta^2}}{2}$$

$$\Rightarrow \Delta = \frac{\delta^2}{2} \pm \delta\sqrt{1+\frac{\delta^2}{4}}$$

Relation between Δ and δ.

$$\Delta = \frac{\delta^2}{2} \pm \delta\sqrt{1+\frac{\delta^2}{4}}$$

where δ and Δ are the central difference and forward difference operator.

Proof: $\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$

$$\Rightarrow \delta f(x) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

Squaring, we get $\delta^2 = E + E^{-1} - 2$

$$E\delta^2 = E^2 + 1 - 2E$$

$$= (E-1)^2$$

$$(1+\Delta)\delta^2 = \Delta^2$$

$$\Delta^2 - \delta^2\Delta - \delta^2 = 0$$

So $\Delta = \frac{\delta^2 \pm \sqrt{\delta^4 + 4\delta^2}}{2}$

$$\boxed{\Rightarrow \Delta = \frac{\delta^2}{2} \pm \delta\sqrt{1+\frac{\delta^2}{4}}}$$

Relation between Δ, ∇, μ and δ: $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

Solution: we know that

$$\delta f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right]$$

$$\Rightarrow \frac{1}{2}\left[E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)\right]$$

$$\mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]$$

But $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

On multiplying μ and δ, we get

$$\mu\delta = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]\left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left[E - E^{-1}\right]$$

$$= \frac{1}{2}[1 + \Delta + \nabla - 1] = \frac{1}{2}[\Delta + \nabla]$$

Relation between D and Δ.

$$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 + \ldots\ldots\ldots\right]$$

Solution: $Ef(x) = f(x+h)$

$$= \left[f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots\ldots\ldots\right]$$

$$= \left[f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \ldots\ldots\ldots\ldots\right]$$

$$= \left[1 + hD + \frac{h^2 D^2}{2} + \ldots\ldots\ldots\ldots\right] f(x)$$

$$Ef(x) = e^{hD} f(x) \quad \left[e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\right]$$

$$E = e^{hD}$$

$$\log E = \log e^{hD}$$
$$\log(1+\Delta) = hD \ \log e^{e}$$
$$\log(1+\Delta) = hD$$
$$D = \frac{1}{h}\log(1+\Delta)$$
$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} + ...\right]$$

$$\left[\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ...\right]$$

Relation between $\Delta, \nabla, \mu \ , \delta$ **and** D **.**

$$hD = \log(1+\Delta) = -\log(1-\nabla) = \sin \ \ h^{-1}(\mu\delta)$$

where D represents the differential operator.

Solution: $Ef(x) = f(x+h)$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ...$$
$$= \left[1 + hD + \frac{h^2D^2}{2!} + ...\right]f(x)$$
$$Ef(x) = e^{hD}f(x)$$
$$\Rightarrow E = e^{hD}$$
$$\Rightarrow \log E = hD \Rightarrow hD = \log(1+\Delta)$$
$$\Rightarrow hD = -\log E^{-1} \Rightarrow hD = -\log(1-\nabla)$$

We know that $\mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]$ & $\delta\left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right]$

$$\Rightarrow \mu\delta = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]\left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right] = \frac{1}{2}(E - E^{-1})$$
$$= \frac{1}{2}\left(e^{hD} - e^{-hD}\right) = SinhhD$$
$$hD = Sinh^{-1}(\mu\delta)$$

Exercise

1. Evaluate $\Delta^2(3e^x)$.

2. Prove that

i. $\Delta \log f(x) = \log[1 + \frac{\Delta f(x)}{f(x)}]$

ii. $(E^{1/2} + E^{-1/2})(1+\Delta)^{\frac{1}{2}} = 2 + \Delta$ where terms have their usual meanings.

3. Prove that

a. $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

b. $\Delta = \frac{\delta^2}{2} \pm \delta\sqrt{1 + \frac{\delta^2}{4}}$

c. $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

d. $\mu^2 = 1 + \frac{\delta^2}{4}$

4. Form the table for backward difference of the function given below, and evaluate $\nabla^4 f(5)$ for $f(x) = x^3 - 3x^2 - 5x - 7$ where x = 0, 1, 2,3,4,5.