Chapter
3

Interpolation

Let $y = f(x)$ be a function of x. The corresponding values of y for a set of arguments $a, a+h, a+2h$................ $a+nh$ are given by

$y_0 = f(a)$
$y_1 = f(a+h)$
$y_2 = f(a+2h)$
....
....
$y_n = f(a+nh)$

Interpolation is the process of finding the values of y for any intermediate values of x between a and $a+nh$.

Extrapolation: Extrapolation is the process of obtaining the values of y for values of x outside the interval a and $a+nh$.

Assumptions for Interpolation:

1. Function is a polynomial function.
2. There is no sudden jump or fall in the values of function under the given interval of argument.
3. The function is either increasing or decreasing uniformly.

Methods for Interpolation:

(a) **For equal Interval**

1. Newton - Gregory Forward Interpolation formula.
2. Newton's Backward Interpolation formula
3. Stirling formula (Central Difference)

(b) **For Unequal Interval.**

1. Lagranges method
2. Newton's divided difference method.

Newton - Gregory Forward Interpolation:

This formula is applied to the functions that has to interpolation near the beginning of the tabulated values.

We know that $f(a+ph) = \mathrm{E}^p f(a) = (1+\Delta)^p f(a)$

On expanding $(1+\Delta)^p$ by binomial theorem, we get

$$f(a+ph) = \left[1 + p\Delta + \frac{p(p-1)}{2}\Delta^2 + \ldots\ldots\ldots\ldots\ldots\right] f(a)$$

$$\Rightarrow f(a+ph) = f(a) + p\Delta f(a) + \frac{p(p-1)}{2}\Delta^2 f(a) + \ldots\ldots\ldots\ldots$$

This is Newton's Gregory formula of Interpolation

Problem: State the appropriate interpolation is to be used to calculated the value of f (1.75) from the following data and hence evaluates it from the given data-

x	1.7	1.8	1.9	2.0
y(x)	5.474	6.050	6.686	7.389

Solution: Difference table is as under.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1.7	5.474			
1.8	6.050	0.576	0.060	0.007
1.9	6.686	0.636	0.067	
2.0	7.389	0.703		

a+ph = 1.75 *a = 1.7* *h = 0.1*

1.7 + p (0.1) = 1.75 $\Rightarrow$ *p = 0.5*

By Newton's forward Interpolation formula, we have -

$$f(a+ph)=f(a)+p\Delta f(a)+\frac{p(p-1)}{2!}\Delta^2 f(a)+\frac{p(p-1)(p-2)}{3!}\Delta^3 f(a)+\ldots\ldots\ldots..$$

$$f(1.75)=f(1.7)+0.5\Delta f(1.7)+\frac{0.5(0.5-1)}{2!}\Delta^2 f(1.7)+\frac{0.5(0.5-1)(0.5-2)}{3!}\Delta^3 f(1.7)+\ldots\ldots..$$

$$=474+0.5\times(0.576)+\frac{0.5(0.5-1)}{2!}(0.060)+\frac{0.5(0.5-1)(0.5-0.5-2)}{3!}(0.007)+\ldots\ldots$$

Problem: If $u_0=1$, $u_1=0$, $u_2=5$, $u_3=22$, $u_4=57$ Find $u_{0.5}$.

Solution: The difference table is as follows;

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)	Δ^4f(x)
0	1				
1	0	-1	6		
2	5	5	12	6	
3	22	17	18	6	0
4	57	35			

Here a + ph = 0.5   a = 0, h = 1
0 + p×1 = 0.5   p = 0.5

By Newton's Forward interpolation formula, we have

$$f(a+ph)=f(a)+p\Delta f(\mathrm{a})+\frac{p(p-1)}{2!}\Delta^2 f(a)+\frac{p(p-1)(p-2)}{3!}\Delta^3 f(a)+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f(a)$$

$$u_{0.5}=u_0+0.5\Delta u_0+\frac{0.5(0.5-1)}{2!}\Delta^2 u_0+\frac{0.5(0.5-1)(0.5-2)}{3!}\Delta^3 u_0+\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}\times 0$$

$$=0+(0.5)(1)+\frac{(0.5)(-0.5)}{2}\times 6+\frac{(0.5)(-0.5)(-1.5)}{6}\times 6+0$$

$$=1-0.5-0.75+0.375=0.125$$

Missing term method:

Problem: Estimate the missing term in the following data.

x	0	1	2	3	4
f(x)	1	3	9	?	81

Solution: Here, four entries are given so y can be represented by third degree polynomial.

Hence

$$\Delta^3 y = Cons\tan t$$

$$\Delta^4 y = 0$$

$$(E-1)^4 y = 0$$

$$(E^4 - 4E^3 + 6E^2 - 4E + 1)y = 0$$

$$E^4 y - 4E^3 y + 6E^2 y - 4Ey + y = 0$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$or\ f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$$

$$81 - 4y_3 + 54 - 12 + 1 = 0$$

$$4y_3 = 124$$

$$y_3 = \frac{124}{4} = 31$$

Problem: Find the missing values in the following table:

x	45	50	55	60	65
y	3	-	2	-	-2.4

Solution: Here, three entries are given so f(x) can be represented by two degree polynomial.

$$\Delta^2 f(x) = Cons\tan t$$

$$\Delta^3 f(x) = 0$$

$$(E-1)^3 f(x) = 0$$

$$(E^3 - 3E^2 + 3E - 1) f(x) = 0$$

$$E^3 f(x) - E^2 f(x) + 3Ef(x) - f(x) = 0 ....(1)$$
$$\Rightarrow f(60) - 3f(55) + 3f(50) - f(45) = 0$$
$$f(60) - 3 \times 2 + 3f(50) - 3 = 0$$
$$f(60) + 3f(50) = 9 ....(2)$$

Again from (1) *f(65) - 3f(60)+3f(55)-f(50) = 0*

- 2.4 - 3f(60) + 3×2 - f(50) = 0

3f(60) + f(50) = 3.6 .... (3)

From (2) and (3) we have

f (60) = 0.225, f(50) = 2.925

Newton's Backward Interpolation formula:

$$y_p = f(x_n + ph) = E^p f(x_n) = (1-\nabla)^{-p}\, y_n \qquad \left[\because E^{-1} = 1-\nabla\right]$$
$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + ...\right] y_n$$
$$y_p = y_n + \nabla y_n + \frac{p(p+1)}{2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3}\nabla^3 y_n + ...$$

is called Newton's Backward interpolation formula.

Remark: (1) This formula is used for finding the value of y for x, when x is near x_n (end).

(2) It is also used for extrapolating values of y for x when x is slightly greater than x_n.

Problem: Given

x	1	2	3	4	5	6	7	8
f(x)	1	8	27	64	125	216	343	512

Estimate f (7.5)?

Solution: Difference table is as follows

x	f(x)	$\nabla f(x)$	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$	$\nabla^6 f$	$\nabla^7 f$
1	1							
2	8	7	12					
3	27	19	18	6	0			
4	64	34	24	6	0	0	0	
5	125	61	30	6	0	0	0	0
6	216	91	36	6	0	0		
7	343	127	42	6				
8	512	169		6				

Here x=7.5, $x_n = 8$

$$x = x_n + ph$$

$$7.5 = 8 + p(1)$$

$$p = - 0.5$$

By Newton's Backward difference formula, we have

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + ...$$

$$= 512 + (-0.5)169 + \frac{(-0.5)(-0.5+1)}{2!}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!} \times 6 + 0$$

$$= 512 - (0.5)169 - (0.5)(0.5) \times 21 - (0.5)(0.5)(1.5)$$

$$= 512 - 84 - 5.25 - 0.375 = 421.875$$

Central Difference:

For Interpolation near the middle of table, we apply central difference formula.

x	y	Ist Diff.	2nd Diff.	3rd Diff.	4th Diff.
x_0-2h	y_{-2}	$\Delta y_{-2} = \delta y_{-3/2}$	$\Delta^2 y_{-2} = \delta^2 y_{-1}$	$\Delta^3 y_{-2} = \delta^3 y_{-1/2}$	
x_0-h	y_{-1}	$\Delta y_{-1} = \delta y_{-1/2}$	$\Delta^2 y_{-1} = \delta^2 y_0$		$\Delta^4 y_{-2} = \delta^4 y_0$
x_0	y_0	$\Delta y_0 = \delta y_{1/2}$	$\Delta^2 y_0 = \delta^2 y_1$	$\Delta^3 y_{-1} = \delta^3 y_{1/2}$	
x_0+h	y_1	$\Delta y_1 = \delta y_{3/2}$			
x_0+2h	y_2				

Stirling Formula (Central Difference)

$$y_n = y_0 + p\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\mu\delta^4 y_0$$

$$+\frac{p(p^2-1^2)(p^2-2^2)}{5!}\mu\delta^5 y_0 + \ldots\ldots\ldots\ldots$$

Problem: Use stirling formula, to find y for x = 35, from the following table.

x	y
20	512
30	439
40	346
50	243

Solution: a + ph = 35 a = 30, h=10

30 + p (10) = 35, p = 0.5

Difference Table:

x	p	y	δ	δ^2	δ^3
20	-1	512			
30	0	439	-73	20	
40	1	346	-93	-10	10
50	2	243	-103		

By stirling formula.

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2}\delta^2 y_0 + \frac{p(p^2-1^2)}{3}\mu\delta^3 y_0 + \ldots\ldots\ldots\ldots$$

$$y_{35} = 439 + (0.5)\times\frac{(-73-93)}{2} + \frac{(0.5^2)}{2}(-20)$$

$$= 439 - (0.5)(83) - 2.5$$

$$= 439 - 41.5 - 2.5$$

$$= 395$$

Interpolation with unequal Interval.

Lagrange's Interpolation formula:

Consider a function f(x). The corresponding value of *f(x)* for x_1, x_2, x_3 .... are $f(x_1)$, $f(x_2)$, $f(x_3)$........ then the Lagranges interpolation formula, to find the value of f(x) for any value of x is

$$f(x)=\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1)+\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$
$$+\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3)+\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)+.....(1)$$

Problem: Using Lagrange's interpolation formula, find the value of y corresponding to x=10 from the following table,

x	5	6	9	11
y	12	13	14	16

Solution: Here $x = 10, x_1=5, x_2=6, x_3=9, x_4=11$
$f(x_1)=12, f(x_2)=13, f(x_3)=14, f(x_4)=16$

Putting the values in (1), we get

$$f(10)=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\times 12+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times 13$$
$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\times 14+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times 16$$
$$=2-\frac{13}{3}+\frac{35}{3}+\frac{16}{3}=14\frac{2}{3}$$

Hence value of y is $14\frac{2}{3}$ corresponding to x=10.

Divided Differences:

Let $f(x_0), f(x_1), f(x_2)$ ………… $f(x_n)$ be the values of the function $y=f(x)$ corresponding to the values x_0, x_1, x_2 ………….. x_n of the argument x.

The Ist divided difference is

$$\underset{x_1}{\Delta} f(x_0) = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\underset{x_2}{\Delta} f(x_1) = f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\underset{x_3}{\Delta} f(x_2) = f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

....

....

$$\underset{x_n}{\Delta} f(x_{n-1}) = f(x_n, x_{n-1}) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Second divided difference.

$$\underset{x_1 x_2}{\Delta} f(x_0) = f(x_0, x_1, x_2) = \frac{f(x_1 x_2) - f(x_0, x_2)}{x_2 - x_0}$$

$$\underset{x_2 x_3}{\Delta} f(x_1) = f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

....

....

$$\underset{x_{n-1} x_n}{\Delta} f(x_{n-2}) = f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-2}}$$

Third divided Difference

$$\underset{x_1, x_2, x_3}{\Delta} f(x_0) = f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$\underset{x_2, x_3, x_4}{\Delta} f(x_1) = f(x_1, x_2, x_3, x_4) = \frac{f(x_2, x_3, x_4) - f(x_1, x_2, x_3)}{x_4 - x_1}$$

Similarly n[th] divided difference is given by.

$$\underset{x_1, x_2 - x_n}{\Delta} f(x_0) = f(x_0, x_1, x_2 ......x_n) = \frac{f(x_1, x_2 ....x_n) - f(x_0, x_1 ....x_{n-1})}{x_n - x_0}$$

Divided Difference Table:

x	***f(x)***	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$			
x_1	$f(x_1)$	$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$	$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$	
x_2	$f(x_2)$	$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$	$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_3)}{x_3 - x_1}$	$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$
x_3	$f(x_3)$	$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$		

Problem: Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

Solution:

x	***f(x)***	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
4	56	$\frac{56-4}{4-2} = 26$	$\frac{131-26}{9-2} = 15$	
9	711	$\frac{711-56}{9-4} = 131$	$\frac{269-131}{10-4} = 23$	$\frac{23-15}{10-2} = 1$
10	980	$\frac{980-711}{10-9} = 269$		

So third divided difference of the given function is 1.

Relation between divided differences and Forward differences:

We prove that:

$$\underset{x_1}{\Delta} f(x_0) = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h}$$

$$so \quad \underset{x_1}{\Delta} f(x_0) = \frac{\Delta f(x_0)}{h}$$

$$\underset{x_2}{\Delta^2} f(x_0) = \frac{\Delta f(x_1) - \Delta f(x_0)}{x_2 - x_0} = \frac{\frac{\Delta f(x_1)}{h} - \frac{\Delta f(x_0)}{h}}{(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{\frac{\Delta f(x_1)}{h} - \frac{\Delta f(x_0)}{h}}{2h} = \frac{\Delta f(x_1) - \Delta f(x_0)}{2h^2} = \frac{\Delta^2 f(x_0)}{2h^2}$$

$$so \quad \underset{x_2}{\Delta^2} f(x_0) = \frac{\Delta^2 f(x_0)}{2!h^2}$$

....

$$\Delta^n f(x_0) = \frac{\Delta^n f(x_0)}{n!h^n}$$

Newton's divided difference Interpolation formula for unequal Intervals:

Let $f(x_0), f(x_1)$ .... $f(x_n)$ be the values of the function $f(x)$ for the corresponding arguments $x_0, x_1,$ .... x_n then the Newton's divided difference formula is

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots\dots\dots\dots \\ &+ (x - x_0)(x - x_1) \dots\dots\dots\dots (x - x_{n-1}) f(x_0, x_1, x_2 \dots\dots\dots x_n) \end{aligned}$$

Problem: Use Newton's divided difference formula to calculate *f(x)* for the following table hence find value for *f (3)*.

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Solution: Newton's divided difference table is

x	***f(x)***	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	1					
1	14	$\frac{14-1}{1-0}=13$	$\frac{1-13}{2-0}=-6$			
2	15				0	
4	5	$\frac{15-14}{2-1}=1$	$\frac{-5-1}{4-1}=-2$	$\frac{-2+6}{4-0}=1$	0	0
5	6					
6	19	$\frac{5-15}{4-2}=-5$	$\frac{1+5}{5-2}=2$	$\frac{2+2}{5-1}=1$		
		$\frac{6-5}{5-4}=1$	$\frac{13-1}{6-4}=6$	$\frac{6-2}{6-2}=1$		
		$\frac{19-6}{6-5}=13$				

By Newton's divided difference interpolation formula

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+\,(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+\ldots\ldots\ldots\ldots\ldots\ldots$$

Putting the values, we have

$$\begin{aligned} f(x) &= 1+(x-0)\times 13+(x-0)(-6)+(x-0)(x-1)(x-2)\times 1 \\ &= 1+13x-6x^2+6x+x^3-3x^2+2x \\ &= x^3-9x^2+21x+1 \end{aligned}$$

$$\begin{aligned} f(3) &= 3^3-9\times 3^2+21\times 3+1 \\ &= 10 \end{aligned}$$

Exercise

1. Use Lagrange's formula to find a polynomial which takes the values $f(0) = -12, f(1) = 0, f(3) = 6$ and $f(4) = 12$. Hence find $f(2)$.

2. Find the function $f(x)$ from the following table using Newton's divided difference formula:

x:	0	1	2	4	5	7
$f(x)$:	0	0	–12	0	600	7308

3. Given the following table, find the number of students whose weight is between 60 and 70 lbs:

Weight (in lbs) x:	0 – 40	40 – 60	60 – 80	80 – 100	100 – 120
No. of students:	250	120	100	70	50

4. Form the divided difference table for the data $(0, 1)$, $(1, 4)$, $(3, 40)$ and $(4, 85)$.

5. From the table, estimate the number of students who obtained marks between 70 and 75.

Marks:	30-40	40-50	50-60	60-70	70-80
No. of Students:	31	42	51	35	31

6. From the following table find the first derivative at x = 4 using Newton's divide difference formula

X:	1	2	4	8	10
F(x):	0	1	5	21	27

7. Compute f(27) from the following data using Lagrange,s interpolation formula.

X:	14	17	31	35
F(x):	68.7	64.0	44.0	39.1