## Exercise

4. Solve $y' = y - \frac{2x}{y}$, y (0) =1, h =.1 for $0 \le x \le .2$

Using (i) Euler's method (ii) Improved Euler's method

Apply the Euler method to approximate the indicated value of the solution function.

5. $y' = x+y$, y(0) =1, Find y(1) ,using h=.1

6. $y' = 1-y$, y(0) =0, Find y(.3), using h=.1

7. $y' = x^3+y$, y(0) =1. Find y(0.02) , using h=.01

8. $y' = x^2+y$, y(0) = 1, Find y (0.02),using h = .01

Apply the improved Euler method to approximate the indicated value of the solution function in following problems.

9. $y' = x^2+y$, y(0) =1, Find y(0.02), using h = .01

10. $y' = x+y$, y(0)=1, Find y(0.3), using h=.1

11. $y' = x+y^2$, y(0) =1, Find y(0.5), using h=.1

Given the initial-value problems, use the Runge Kutta method with h = 0.1 to obtain four decimal-place approximation to the indicated value.

12. $y' = x^2-y$, y(0) =1; y(0.1), y(0.2)

13. $y' = x^2+y^2$, y(1) = 1.5 ; y(1.2)

14. $y' = x+y^2$, y(0) =1; y(0.2)

# Chapter
# 7

## System of Linear Equation

**Crouts - Triangularisation Method Direct Method to Solve system of Linear equation.** This method is based on the fact that every square matrix A is the product of a lower triangular matrix and an upper triangular matrix.

**Method:** Consider the following equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These equations are written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{........................(1)}$$

AX=B

Now, let A = LU

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{.....(2)}$$

Multiplying the matrices on R.H.S., we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Equating corresponding elements on both sides, we get

$$l_{11} = a_{11} \qquad l_{11}u_{12} = a_{12} \qquad l_{11}u_{13} = a_{13}$$
$$l_{21} = a_{21} \qquad l_{21}u_{12} + l_{22} = a_{22} \qquad l_{21}u_{13} + l_{23}u_{23} = a_{23}$$
$$l_{31} = a_{31} \qquad l_{31}u_{12} + l_{32} = a_{32} \qquad l_{31}u_{13} + l_{32}u_{23} + l_{23} = a_{33}$$

We solve these equations in the following order

Step-I Solve equation in Ist column.
Step-II Solve equations in Ist row.
Step-III Solve equations in II column.
Step-IV Solve equations in 2nd row.
Step-V Solve equations in 3rd column.
Step-VI Solve equations in 3rd row.

Putting LU for A in (1), we have

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{........................(3)}$$

Now put $\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ ........................(4)

Than (iv) becomes $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ ........................(5)

On solving equation (5), we get values of u, u, w. we substitute the values of u, v, w in (iv) and on solving (iv), we get values of $x_1, x_2, x_3.$

**Problem:** Apply Crout's method (Factorization method) to solve the equation

$$3x + 2y + 7z = 4,\ 2x + 3y + z = 5,\ 3x + 4y + z = 7$$

**Solution:** We have

$$3x + 2y + 7z = 4$$
$$2x + 3y + z = 5$$
$$3x + 4y + z = 7$$

Which can be written in the matrix form as

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \quad \text{........................(1)}$$

Where $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ ........................(2)

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \quad \text{..................................(3)}$$

Equating corresponding elements on both sides of (2), we get

(1) First row, $u_{11} = 3,\ u_{13} = 2,\ u_{13} = 7$

(2) First Column, $l_{21}u_{11} = 2 \Rightarrow l_{21} = \frac{2}{3}$ $\quad l_{31}u_{11} = 3 \Rightarrow l_{31} = \frac{3}{3} = 1$

(3) Second Row

$$l_{21}u_{12}+u_{22}=3 \Rightarrow u_{22}=3-\frac{2}{3}\times 2=3-\frac{4}{3}=\frac{5}{3}$$

$$l_{21}u_{13}+u_{23}=1 \Rightarrow u_{23}=1-\frac{2}{3}\times 7=1-\frac{14}{3}=-\frac{11}{3}$$

(4) Second Column,

$$l_{31}u_{12}+l_{32}u_{22}=4 \Rightarrow 1\times 2+l_{32}\times\frac{5}{3}=4$$

$$\Rightarrow l_{32}=\frac{3}{5}(4-2)=\frac{6}{5}$$

(5) Third Row

$$l_{31}u_{13}+l_{32}u_{23}+u_{33}=1$$

$$\Rightarrow 1\times 7+\frac{6}{5}\times\frac{-11}{3}+u_{33}=1 \Rightarrow u_{33}=1-7+\frac{66}{15}=-\frac{8}{5}$$

Putting the values in (1) we get

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 1 & \frac{6}{5} & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 7 \\ 0 & \frac{5}{3} & -\frac{11}{3} \\ 0 & 0 & -\frac{8}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \text{........................(4)}$$

Writing $\begin{bmatrix} 3 & 2 & 7 \\ 0 & \frac{5}{3} & -\frac{11}{3} \\ 0 & 0 & -\frac{8}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ ........................(5)

(4) becomes $\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 1 & \frac{6}{5} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ \frac{2}{3}u + v \\ u + \frac{6}{5}v + w \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

$so\;\; u = 4,\; \frac{2}{3}u + v = 5 \Rightarrow v = 5 - \frac{8}{3} = \frac{7}{3}$

$u + \frac{6}{5}v + w = 7 \Rightarrow w = 7 - 4 - \frac{42}{15} = \frac{1}{5}$

Putting the values of u, v, w in (5) we get

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & \frac{5}{3} & -\frac{11}{3} \\ 0 & 0 & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{7}{3} \\ \frac{1}{5} \end{bmatrix}$$

i.e. $3x + 2y + 7z = 4$........................... (6)

$\frac{5}{3}y - \frac{11}{8}z = \frac{7}{3}$ .................... (7)

$-\frac{8}{5}z = \frac{1}{5} \Rightarrow z = -\frac{1}{8}$ .................... (8)

Putting $z = -\frac{1}{8}$, in (7) we get $y = \frac{9}{8}$

Putting the values of *z and y* in (6), we get $x = \frac{7}{8}$

Hence the solution of system of Linear equation is

$x = \frac{7}{8},\; y = \frac{9}{8},\; z = -\frac{1}{8}$

**Gauss-Seidel Method.** [Iterative Method]
In this method, we use the value obtained in the earlier step.

Let us consider system of linear equation

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The above equations can be written as

$x = c_1 - k_{12}y - k_{13}z$......................... *(1)*
$y = c_2 - k_{21}x - k_{23}z$....................... *(2)*
$z = c_3 - k_{31}x - k_{32}y$....................... *(3)*

**Step-I:** First we put $y = z = 0$ in (1) and get $x = c_1$.
Then in (2) equation we put value of $x$ i.e. $c_1$ and $z=0$ and obtain $y$. In third equation we use the value of $x$ and $y$ obtained earlier to get $z$.

**Step-II.** We repeat the above procedure.
In other words, latest values of the unknown are used in each step.

**Problem:** Solve the following system of linear equation by Gauss Siedel method.

$$23x_1 + 13x_2 + 3x_3 = 29$$
$$5x_1 + 23x_2 + 7x_3 = 37$$
$$11x_1 + x_2 + 23x_3 = 43$$

**Solution:** Here, we have

$$23x_1 + 13x_2 + 3x_3 = 29$$
$$5x_1 + 23x_2 + 7x_3 = 37$$
$$11x_1 + x_2 + 23x_3 = 43$$

Solving each equation of the given system for the unknowns with largest coefficient in terms of the remaining unknowns, we have,

$$x_1 = \frac{1}{23}\left(29 - 13x_2 - 3x_3\right) \text{ ..................... (1)}$$

$$x_2 = \frac{1}{23}\left(37 - 5x_1 - 7x_3\right) \text{ ..................... (2)}$$

$$x_3 = \frac{1}{23}\left(43 - 11x_1 - x_2\right) \text{ ..................... (3)}$$

**For Ist Iteration:**

Putting $x_2 = 0, x_3 = 0$ in (1), we get

$$x_1 = \frac{1}{23}\left(29\right) = 1.2608$$

Putting $x_1 = 1.26087, x_3 = 0$ in (2), we get

$$x_2 = \frac{1}{23}\left(37 - 5 \times 1.26087 - 0\right) = 1.33459$$

Putting $x_1 = 1.26087$ *and* $x_2 = 1.33459$ in (3) we get

$$x_3 = \frac{1}{23}[\left(43 - 11 \times 1.26087 - 1.33459\right)]$$

$$x_3 = \frac{1}{23}\left[43 - 13.86957 - 1.33459\right]$$

$$= 1.20851$$

**For Second Iteration:**

Putting $x_2 = 1.33459$ *and* $x_3 = 1.20851$ in (1), we get

$$x_1 = \frac{1}{23}\left(29 - 13 \times 1.33459 - 3 \times 1.20851\right)$$

$$= \frac{1}{23}\left(29 - 17.34967 - 3.62553\right) = 0.34890$$

Putting $x_1 = 0.34890$ *and* $x_3 = 1.20851$ in (2), we get

$$x_2 = \frac{1}{23}\left(37 - 5 \times 0.34890 - 7 \times 1.20851\right)$$

$$= \frac{1}{23}\left(37 - 1.74450 - 8.45957\right) = 1.16504$$

Putting $x_1 = 0.34890$ *and* $x_2 = 1.6504$ in (3), we get

$$x_3 = \frac{1}{23}[43 - 11 \times 0.34890 - 1.16504]$$

$$= \frac{1}{23}[43 - 3.8379 - 1.16504] = 1.65205$$

**For Third Iteration**

Putting $x_2 = 1.16504$ *and* $x_3 = 1.65205$ in (1), we get

$$x_1 = \frac{1}{23}[29 - 13 \times 1.16504 - 3 \times 1.65205]$$

$$= \frac{1}{23}[29 - 15.1502 - 4.95615] = 0.38668$$

Putting $x_1 = 0.38668$ *and* $x_3 = 1.65205$ in (2), we get

$$x_2 = \frac{1}{23}[37 - 5 \times 0.38668 - 7 \times 1.65205]$$

$$= \frac{1}{23}[37 - 1.9334 - 11.56435] = 1.02184$$

Putting $x_1 = 0.38668$ *and* $x_2 = 1.02184$ in (3), we get

$$x_3 = \frac{1}{23}[43 - 11 \times 0.38668 - 1.02184]$$

$$= \frac{1}{23}[43 - 4.25348 - 1.02184] = 1.640201$$

## Exercise

1. Use Gauss – Seidal iterative method to obtain the solution of the equations:
$9x - y + 2z = 9;\ x + 10y - 2z = 15;\ 2x - 2y - 13z = -17$ .

2. Solve the following system of equations using Gauss-Seidel iteration method:
$2x + 10y + z = 51$, $10x + y + 2z = 44$, $x + 2y + 10z = 61$.

3. Solve the given system of equations using Crout's method.
3x + 6y + 9z =-12, -x + 2y + z = 20, 2x –3y +10z = 3

4. Solve the given system of equations using Crout's
5x + 2y + z =-12, -x + 4y + 2z =20, 2x–3y +10z =3.