

# Lecture -7

## (System of Linear Equation)

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# Chapter 7

## System of Linear Equation

**Crouts - Triangularisation Method Direct Method to Solve system of Linear equation.** This method is based on the fact that every square matrix A is the product of a lower triangular matrix and an upper triangular matrix.

**Method:** Consider the following equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These equations are written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \dots\dots\dots(1)$$

$$AX=B$$

Now, let

$$A = LU$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \dots\dots(2)$$

Multiplying the matrices on R.H.S., we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Equating corresponding elements on both sides, we get

$$\begin{matrix} l_{11} = a_{11} & l_{11}u_{12} = a_{12} & l_{11}u_{13} = a_{13} \\ l_{21} = a_{21} & l_{21}u_{12} + l_{22} = a_{22} & l_{21}u_{13} + l_{23}u_{23} = a_{23} \\ l_{31} = a_{31} & l_{31}u_{12} + l_{32} = a_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{23} = a_{33} \end{matrix}$$

We solve these equations in the following order

- Step-I        Solve equation in Ist column.
- Step-II       Solve equations in Ist row.
- Step-III      Solve equations in II column.
- Step-IV      Solve equations in 2nd row.
- Step-V        Solve equations in 3rd column.
- Step-VI      Solve equations in 3rd row.

Putting LU for A in (1), we have

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \dots\dots\dots(3)$$

Now put  $\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \dots\dots\dots(4)$

Than (iv) becomes  $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \dots\dots\dots(5)$

On solving equation (5), we get values of u, u, w. we substitute the values of u, v, w in (iv) and on solving (iv), we get values of  $x_1, x_2, x_3$ .

**Problem:** Apply Crout's method (Factorization method) to solve the equation

$$3x + 2y + 7z = 4, \quad 2x + 3y + z = 5, \quad 3x + 4y + z = 7$$

**Solution:** We have

$$3x + 2y + 7z = 4$$

$$2x + 3y + z = 5$$

$$3x + 4y + z = 7$$

Which can be written in the matrix form as

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \dots\dots\dots(1)$$

Where  $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \dots\dots\dots(2)$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \dots\dots\dots(3)$$

Equating corresponding elements on both sides of (2), we get

(1) First row,  $u_{11} = 3, \quad u_{12} = 2, \quad u_{13} = 7$

(2) First Column,  $l_{21}u_{11} = 2 \Rightarrow l_{21} = \frac{2}{3} \quad l_{31}u_{11} = 3 \Rightarrow l_{31} = \frac{3}{3} = 1$

(3) Second Row

$$l_{21}u_{12} + u_{22} = 3 \Rightarrow u_{22} = 3 - \frac{2}{3} \times 2 = 3 - \frac{4}{3} = \frac{5}{3}$$

$$l_{21}u_{13} + u_{23} = 1 \Rightarrow u_{23} = 1 - \frac{2}{3} \times 7 = 1 - \frac{14}{3} = -\frac{11}{3}$$

(4) Second Column,

$$l_{31}u_{12} + l_{32}u_{22} = 4 \Rightarrow 1 \times 2 + l_{32} \times \frac{5}{3} = 4$$

$$\Rightarrow l_{32} = \frac{3}{5}(4 - 2) = \frac{6}{5}$$

(5) Third Row

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$$

$$\Rightarrow 1 \times 7 + \frac{6}{5} \times \frac{-11}{3} + u_{33} = 1 \Rightarrow u_{33} = 1 - 7 + \frac{66}{15} = -\frac{8}{5}$$

Putting the values in (1) we get

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 1 & \frac{6}{5} & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & \frac{5}{3} & -\frac{11}{3} \\ 0 & 0 & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \dots\dots\dots(4)$$

Writing  $\begin{bmatrix} 3 & 2 & 7 \\ 0 & \frac{5}{3} & -\frac{11}{3} \\ 0 & 0 & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \dots\dots\dots(5)$

$$(4) \text{ becomes } \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 1 & \frac{6}{5} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ \frac{2}{3}u + v \\ u + \frac{6}{5}v + w \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

$$\text{so } u = 4, \frac{2}{3}u + v = 5 \Rightarrow v = 5 - \frac{8}{3} = \frac{7}{3}$$

$$u + \frac{6}{5}v + w = 7 \Rightarrow w = 7 - 4 - \frac{42}{15} = \frac{1}{5}$$

Putting the values of  $u, v, w$  in (5) we get

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & \frac{5}{3} & -\frac{11}{3} \\ 0 & 0 & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{7}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\text{i.e. } 3x + 2y + 7z = 4 \dots\dots\dots (6)$$

$$\frac{5}{3}y - \frac{11}{8}z = \frac{7}{3} \dots\dots\dots (7)$$

$$-\frac{8}{5}z = \frac{1}{5} \Rightarrow z = -\frac{1}{8} \dots\dots\dots (8)$$

Putting  $z = -\frac{1}{8}$ , in (7) we get  $y = \frac{9}{8}$

Putting the values of  $z$  and  $y$  in (6), we get  $x = \frac{7}{8}$

Hence the solution of system of Linear equation is

$$x = \frac{7}{8}, y = \frac{9}{8}, z = -\frac{1}{8}$$

**Gauss-Seidel Method.** [Iterative Method]

In this method, we use the value obtained in the earlier step.

Let us consider system of linear equation

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The above equations can be written as

$$x = c_1 - k_{12}y - k_{13}z \dots \dots \dots (1)$$

$$y = c_2 - k_{21}x - k_{23}z \dots \dots \dots (2)$$

$$z = c_3 - k_{31}x - k_{32}y \dots \dots \dots (3)$$

**Step-I:** First we put  $y = z = 0$  in (1) and get  $x = c_1$ .

Then in (2) equation we put value of  $x$  i.e.  $c_1$  and  $z=0$  and obtain  $y$ . In third equation we use the value of  $x$  and  $y$  obtained earlier to get  $z$ .

**Step-II.** We repeat the above procedure.

In other words, latest values of the unknown are used in each step.

**Problem:** Solve the following system of linear equation by Gauss Siedel method.

$$23x_1 + 13x_2 + 3x_3 = 29$$

$$5x_1 + 23x_2 + 7x_3 = 37$$

$$11x_1 + x_2 + 23x_3 = 43$$

**Solution:** Here, we have

$$23x_1 + 13x_2 + 3x_3 = 29$$

$$5x_1 + 23x_2 + 7x_3 = 37$$

$$11x_1 + x_2 + 23x_3 = 43$$

Solving each equation of the given system for the unknowns with largest coefficient in terms of the remaining unknowns, we have,

$$x_1 = \frac{1}{23}(29 - 13x_2 - 3x_3) \dots\dots\dots (1)$$

$$x_2 = \frac{1}{23}(37 - 5x_1 - 7x_3) \dots\dots\dots (2)$$

$$x_3 = \frac{1}{23}(43 - 11x_1 - x_2) \dots\dots\dots (3)$$

**For 1st Iteration:**

Putting  $x_2 = 0$ ,  $x_3 = 0$  in (1), we get

$$x_1 = \frac{1}{23}(29) = 1.2608$$

Putting  $x_1 = 1.26087$ ,  $x_3 = 0$  in (2), we get

$$x_2 = \frac{1}{23}(37 - 5 \times 1.26087 - 0) = 1.33459$$

Putting  $x_1 = 1.26087$  and  $x_2 = 1.33459$  in (3) we get

$$x_3 = \frac{1}{23}[(43 - 11 \times 1.26087 - 1.33459)]$$

$$x_3 = \frac{1}{23}[43 - 13.86957 - 1.33459]$$

$$= 1.20851$$

**For Second Iteration:**

Putting  $x_2 = 1.33459$  and  $x_3 = 1.20851$  in (1), we get

$$x_1 = \frac{1}{23}(29 - 13 \times 1.33459 - 3 \times 1.20851)$$

$$= \frac{1}{23}(29 - 17.34967 - 3.62553) = 0.34890$$

Putting  $x_1 = 0.34890$  and  $x_3 = 1.20851$  in (2), we get

$$x_2 = \frac{1}{23}(37 - 5 \times 0.34890 - 7 \times 1.20851)$$

$$= \frac{1}{23}(37 - 1.74450 - 8.45957) = 1.16504$$



Putting  $x_1 = 0.34890$  and  $x_2 = 1.6504$  in (3), we get

$$\begin{aligned} x_3 &= \frac{1}{23}[43 - 11 \times 0.34890 - 1.16504] \\ &= \frac{1}{23}[43 - 3.8379 - 1.16504] = 1.65205 \end{aligned}$$

### For Third Iteration

Putting  $x_2 = 1.6504$  and  $x_3 = 1.65205$  in (1), we get

$$\begin{aligned} x_1 &= \frac{1}{23}[29 - 13 \times 1.6504 - 3 \times 1.65205] \\ &= \frac{1}{23}[29 - 15.1502 - 4.95615] = 0.38668 \end{aligned}$$

Putting  $x_1 = 0.38668$  and  $x_3 = 1.65205$  in (2), we get

$$\begin{aligned} x_2 &= \frac{1}{23}[37 - 5 \times 0.38668 - 7 \times 1.65205] \\ &= \frac{1}{23}[37 - 1.9334 - 11.56435] = 1.02184 \end{aligned}$$

Putting  $x_1 = 0.38668$  and  $x_2 = 1.02184$  in (3), we get

$$\begin{aligned} x_3 &= \frac{1}{23}[43 - 11 \times 0.38668 - 1.02184] \\ &= \frac{1}{23}[43 - 4.25348 - 1.02184] = 1.640201 \end{aligned}$$

### Exercise

1. Use Gauss – Seidal iterative method to obtain the solution of the equations:

$$9x - y + 2z = 9; \quad x + 10y - 2z = 15; \quad 2x - 2y - 13z = -17.$$

2. Solve the following system of equations using Gauss-Seidel iteration method:

$$2x + 10y + z = 51, \quad 10x + y + 2z = 44, \quad x + 2y + 10z = 61.$$

3. Solve the given system of equations using Crout's method.

$$3x + 6y + 9z = -12, \quad -x + 2y + z = 20, \quad 2x - 3y + 10z = 3$$

4. Solve the given system of equations using Crout's

$$5x + 2y + z = -12, \quad -x + 4y + 2z = 20, \quad 2x - 3y + 10z = 3.$$