RANK OF A MATRIX.

Let A be any $m \times n$ matrix. It has square sub-matrices of different orders. The determinants of these square sub-matrices are called minors of A.

A matrix A is said to be of rank $r\ if$

(i) It has atleast on non-zero minor of order r.

(ii) All the minors of order $(r-1)$ or higher than r are zero.

Symbolically, rank of $A = r$ is written as $\rho(A) = r$.

Form the definition of the rank of a matrix A, it follows that:

(i) If A is a null matrix, then $\rho(A) = 0$. [$\because$ every minor of A has zero value.]

(ii) If A is not a null matrix, then $\rho(A) \geq 1$.

(iii) If A a non-singular $n \times n$ matrix, then $\rho(A) = n$.

[$\because$ $|A| \neq 0$ is the largest minor of A.]

If I_n is the $n \times n$ unit matrix, then $|I_n| = 1 \neq 0 \Rightarrow \rho(I_n) = n$.

(iv) If A is an $m \times n$ matrix, then $\rho(A) \leq$ minimum of m and n.

(v) If all minors of order r are equal to zero, then $\rho(A) < r$.

METHODS OF RINDING RANK

To determine the rank of a matrix A, we adopt the following different methods.

(1) Start with the highest order minor (or minors of A. Let their order be r. If any one of them is non-zero, then $\rho(A) = r$.

If all of them are zero, start with minors of next lower order $(r-1)$ and so on till you get a non-zero minor. The order of that minor is the rank of A.

This method usually involves a lot of computational work since we have to evaluate several determinants.

(2) Normal form method. If A is an $m \times n$ matrix and by a series of elementary (row or column or both) operations, it can be put into one of the following forms (called normal or canonical forms):

$$\begin{bmatrix} I_r & \vdots & O \\ \cdots & \cdots & \cdots \\ O & \vdots & O \end{bmatrix}, \begin{bmatrix} I_t \\ \cdots \\ O \end{bmatrix}, [I_r \ \vdots \ O], [I_r], \text{ where } I_r \text{ is the unit matrix of order } r.$$

Since the rank of a matrix is not changed as a result of elementary transformation, it follows that

$$\rho(A) = r \qquad [\because\ r^{th} \text{ order minor } |I_r| = 1 \neq 0\ ]$$

This method is also called sweep out method or pivotal method.

Procedure to obtain normal form

(i) Interchange rows (or columns) to obtain a non-zero element in I row and I column of given matrix.

(ii) Make this non-zero element as 1.

(iii) Obtain zeros in the remainder of I row and I column.

(iv) Repeat the above three steps starting with element in II row and II column.

(v) Continue the process down the main diagonal either until the end of diagonal is reached or all the remaining elements of matrix are zero.

(3) **Echelon form method**. In this form of the matrix, each of the first r elements of the leading diagonal is non-zero and every element below this diagonal/r^{th} row is zero. A matrix is reduced in Echelon form as

(i) The first non-zero element in a row should be unity (if possible)

(ii) All the non- zero rows, if any, precede the zero rows.

The rank of the matrix is equal to the number of non-zero diagonal elements or the number of non zero rows (as the case may be) when it has been reduced to Echelon form.

In other words, a matrix $A = [a_{ij}]$ is an Echelon matrix or is said to be in Echelon form if the number of zeros preceding the first non-zero entry of a row increases row by row until only zero rows remain.

In row reduced Echelon form (row canonical form) of the matrix, the first non-zero entry of a row is unity and are the only non-zero entry in their respective columns.

Further, any matrix can be transformed to a column Echelon form (CEF) by sequence of elementary column transformations. we any that a matrix is in CEF if

(i) All zero columns are on the right

(ii) The first (if we go down form the top)non-zero entry in each column is 1.

(iii) If $j > i$, then the leading one of the column C_j appears below that of C_i.

$e.g.,$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

A matrix is said to be in a column reduced Echelon form if it is in CEF and additionally, any row containing the leading one of a column consists of all zeros except this leading one. In above examples, first and third matrices are in column reduced echelon form.

FOR AN $m \times n$ MATRIX A, TO FIND SQUARE MATRICES P AND Q OF OR. DERS m AND n RESPECTIVELY, SUCH THAT PAQ IS IN THE NORMAL FORM

Method. Write $A = IAI$

Reduce the matrix on LHS to normal form by affecting elementary row and/or column transformations.

Every elementary row transformation on A must be accompanied by the same transformation on the pre-factor on RHS.

Every elementary column transformation on A must be accompanied by the same transformation on the post-factor on RHS.

RANK-NULLITY THEOREM

If A is an $m \times n$ order matrix over some field then

rank (A) + nul (A) = n

This applies to linear maps also. Let V be a finite dimensional vector space and W be a vector space over some field. Let $T: V \to W$ be a linear map, then

rank (T) + nnul (T) = din (V)

where the rank of T is the dimension of the image of T and nullity of T is the dimension of the kernel of T.

ILLUSTRATIVE EXAMPLES

Example 1. Find the rank of the matrix.

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$

Sol. (i) Here $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$

Operating $R_{21}(2)$

$$- \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 11 & 15 \end{bmatrix}$$

Which is Echelon form.

$\therefore \quad \rho(A) =$ no. of non-zero rows = 2

Example 2. *Use elementary transformations to reduce the following matrix A to triangular form and hence fund the rank of A.*

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Sol. We have.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Operating R_{12}

$$- \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Operating $R_{21}(-2), R_{31}(-3), R_{41}(-6)$

$$- \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Operating $R_{23}(-1)$

$$- \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Operating $R_{32}(-4), R_{42}\ (-9)$

$$- \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

Operating $R_{43}(-2)$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is upper triangular form.

$\therefore \qquad \rho(A) = 3.$

Example 4. (i) *Reduce the matrix A to its normal form when*

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Hence find the rank of A.

(ii) *Reduce the matrix A to its normal form and hence find its rank, where*

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Sol. (i)

Pivot element

$$A = \begin{bmatrix} \boxed{1} & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Operating $R_{21}(-2)R_{31}(-1)R_{41}(1)$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

Operating $R_{21}(-2)R_{31}(1)R_{41}(-4)$

$$- \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

Operating C_{32}

$$- \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}$$

Operating $R_{23}(-1)$

$$\text{Pivot element} \leftarrow - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}$$

Operating $R_{32}(-4), R_{42}(-5)$

$$- \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

Operating $C_{42}(4)$

$$- \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

Operating C_{43}

$$- \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 17 & 0 \end{bmatrix}$$

Operating $R_{34}\left(-\frac{15}{17}\right)$

$$\text{Pivot element} \leftarrow - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 17 & 0 \end{bmatrix}$$

Operating $R_{43}(-17)$

$$\left[\begin{array}{cccc}1 & 0 & 0 & \vdots & 0\\ 0 & 1 & 0 & \vdots & 0\\ 0 & 0 & 1 & \vdots & 0\\ \hdashline 0 & 0 & 0 & \vdots & 0\end{array}\right]$$

$$\sim\left[\begin{array}{ccc}I_3 & \vdots & O\\ \hdashline O & \vdots & O\end{array}\right] \text{ which is normal form.}$$

Hence $\rho(A) = 3$

(ii) $$A = \begin{bmatrix}2 & 1 & -3 & -6\\ 3 & -3 & 1 & 2\\ 1 & 1 & 1 & 2\end{bmatrix}$$

Operating R_{13}

$$\sim\begin{bmatrix}1 & 1 & 1 & 2\\ 3 & -3 & 1 & 2\\ 2 & 1 & -3 & -6\end{bmatrix}$$

Operating $R_{21}(-3), R_{31}(-2)$

$$\sim\begin{bmatrix}1 & 1 & 1 & 2\\ 0 & -6 & -2 & -4\\ 0 & -1 & -3 & -6\end{bmatrix}$$

Operating $C_{21}(-1)C_{31}(-1)C_{41}(-2)$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -6 & -2 & -4\\ 0 & -1 & -5 & -10\end{bmatrix}$$

Operating $R_2(-1)R_3(-1)$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 6 & 2 & 4\\ 0 & 1 & 5 & 10\end{bmatrix}$$

Operating R_{23}

$$-\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 6 & 2 & 4 \end{bmatrix}$$

Operating $R_{32}(-6)$

$$-\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & -28 & -56 \end{bmatrix}$$

Operating $C_{32}(-5), C_{42}(-10)$

$$-\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -28 & -56 \end{bmatrix}$$

Operating $R_3\left(-\frac{1}{28}\right)$

$$-\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Operating $C_{43}(-2)$

$$-\begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

$-[I_3 \ \vdots \ O]$ Which is normal form?

Hence $\rho(A) = 3.$

Example 5. *If* $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, *determine two non- singular matrices P and Q such that* $PAQ = I$. *Hence find* A^{-1}.

Sol. Let $A = I_3 \, A I_3$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Operating $R_{12}(-1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Operating $R_{21}(-2)$

$$\begin{bmatrix}1 & 0 & 0\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

Operating $R_{23}(-4)$

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

Operating $C_{23}(1)$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix},$$

$\Rightarrow \qquad I = PAQ$

where $\qquad P = \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ 0 & 0 & 1\end{bmatrix}$ and $Q = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix},$

Now, $\quad PAQ = I$

Pre-multiplying by P^{-1}

$$P^{-1}PAQ = P^{-1}$$

$$AQ = P^{-1} \qquad \Big|\because\ P^{-1}P = I$$

Post-multiplying by P,

$$AQP = I$$

Per-multiplying by A^{-1},

$$QP = A^{-1}$$

$$A^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{bmatrix},$$

Example 7. *Prove that the points $(x_1, y_1), (x_2, y_2)$ and $(x_3,\ y_3)$ are collinear if and only if the rank of the matrix* $\begin{bmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{bmatrix}$ *is less than three.*

Sol. The condition is necessary

We are given that the points $(x_{i,}\ y_i)$; $\ i = 1, 2, 3$ are collinear. We are to prove that the rank of matrix $\begin{bmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{bmatrix}$ is less than three.

Since the points are collinear, therefore, the area of the triangle formed by these points is zero.

Hence $\frac{1}{2}\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0 \qquad \Rightarrow \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$

$\Rightarrow$ The rank of matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3.

Hence the condition is necessary.

The condition is sufficient.

We are given that the rank of the matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3 and we are to prove that the points $(x_i, \;\; y_i);\; i = 1, 2, 3$ are collinear. Since rank of matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3

hence $$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

$\Rightarrow$ $$\frac{1}{2}\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

Hence the given points are collinear.

$\therefore$ The condition is sufficient.

Example 8. *Show that*

(i) $rank\ (AA') = rank\ (A)$ (ii) $rank\ (A) = rank\ (AA*)$.

Sol. (i) Let $B = AA'$, then

$$\rho(B) = \rho(AA')$$

$$\leq \rho(A) \qquad \Big| \because \text{Rank of a product of two matrices cannot exceed the rank of either matrix}$$

$\therefore$ $\rho(B) \leq \rho(A)$ ... (1)

Now, $B = AA'$

$\Rightarrow$ $A^{-1}B = A'$

$\therefore \quad \rho(A) = \rho(A') = \rho(A^{-1}B) \leq \rho(B)$ | $\therefore$ A mtrix and its transpose have the same rank.

$\therefore \quad \rho(A) \leq \rho(B)$ ...(2)

Form (1) and (2), $\rho(A) = \rho(B) = \rho(AA')$

(ii) Let $C = AA*$

$\therefore \quad \rho(C) = \rho(AA*) \leq \rho(A)$

$\therefore \quad \rho(C) \leq \rho(A)$ ... (1)

Again, $C = AA*$

$\Rightarrow \quad A^{-1}C = A*$

Now, $\rho(A) = \rho(A*) = \rho(A^{-1}C)$ | $\because \quad \rho(A) = \rho(A*)$

$\leq \rho(C)$

$\therefore \quad \rho(A) \leq \rho(C)$ ... (2)

Combining (1) and (2), we get

$$\rho(A) = \rho(C)$$

$\Rightarrow \quad \rho(A) = \rho(AA*)$

TEST YOUR KNOWLEDGE

1. Find the rank of the matrices

(i) $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$

3. (i) Reduce the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix}$ to column echelon form and find its rank.

(ii) Reduce A to Echelon form and then to its row canonical form where

$$A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

Hence find the rank of A.

4. Using elementary transformation, reduce the following matrices to the canonical form.

(i) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 4 & -12 & 8 & 9 \\ 0 & 2 & -6 & 2 & 5 \\ 0 & 1 & -3 & 6 & 4 \\ 0 & -8 & 24 & 3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 3 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

5. (i) Find non-singular matrices P and Q such that PAQ is in the normal form for the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

(ii) Find non- singular matrix P and Q such that PAQ is in normal form of the matrix and hence find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

6. Find the rank of the following matrices using elementary transformations.

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 8 & 11 & 14 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 4 & -3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

7. Find the rank of the following matrices by reducing it to normal form (or canonical form)

(i) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 3 & 2 & 5 & 1 \\ 2 & 2 & -1 & 6 & 3 \\ 1 & 1 & 2 & 3 & -1 \\ 0 & 2 & 5 & 2 & -3 \end{bmatrix}$

8. Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$; a, b, c being all real.

9. Show that the rank of a skew-symmetric matrix cannot be unity.

10. (i) Find all values of μ for which rank of matrix

$$A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$

(ii) Find the value of P for which the matrix

$$A = \begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix} \text{ is of rank 1.}$$

11. Under what condition, the rank of the following matrix A is 3?

Is it possible for the rank to be 1? Why?

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$$

12. If $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 3 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, find rank of A, rank of B, rank of A + B and of AB.

13. Show that if A is a non-zero column matrix and B is a non-zero row matrix then $\rho(AB) = 1$.

14. (i) Show that the rank of the transpose of a matrix is the same as that of the original matrix.

(ii) Prove that for a $m \times n$ matrix, whose every element is 1, the rank is one.

ANSWERS

1. (i) 3 (ii) 3
2. (i) 4
3. (i) 2 (ii) 2
4. (i) $\left[\begin{array}{c:c} I_3 & O_{3\times 2} \\ \hdashline O_{1\times 3} & O_{1\times 2} \end{array}\right]$ (ii) $\left[\begin{array}{c:c} I_3 & O_{3\times 2} \\ O_{1\times 3} & O_{1\times 2} \end{array}\right]$ (iii)) $\left[\begin{array}{c:c} I_3 & O_{3\times 2} \\ O_{1\times 3} & O_{1\times 1} \end{array}\right]$
5. (i) $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{15} & -\frac{1}{21} \\ 0 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}; \ \rho(A) = 2$

6. (i) 2 (ii) 4 (iii) 2 (iv) 3

7. (i) 3 (ii) 2 (iii) 3

8. $\rho(A) = 3$ if $a \neq b \neq c$ and $a + b + c \neq 0$;

$\rho(A) = 2$ if $a \neq b \neq c$ and $a + b + c = 0$

Also $\rho(A) = 2$ if $a = b \neq c$ while $\rho(A) = 1$ if $a = b = c$

10. (i) $\mu = 1, 2, 3$ (ii) $P = 3$

11. $x = \frac{3}{5}; No$

12 $\rho(A) = 3, \rho(B) = 1, \ \rho(A + B) = 3, \ \ \rho(AB) = 1.$