Tutorial-1 B. Tech-I Sem 15 Sep. 2025

Limit and Continuity

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1. Review of some important results which should be remembered.
   (a) Uniqueness of limit If $\lim_{x\to a} f(x) = l$, and $\lim_{x\to a} f(x) = m$, then $l = m$ i.e., $\lim_{x\to a} f(x)$ exists, then it is unique. k_1
   (b) If $\lim_{x\to a} f(x) = l \neq 0$, then these exist a number $k > 0, \delta > 0$ such that $|f(x)| > k$ whenever $0 < |x - a| < \delta$ k_1
   (c) The limit of a sum is equal to the sum of the limits i.e., $\lim_{x\to a}\{(f+g)(x)\} = l + x$ k_1
   (d) The limit of a product is equal to the product of the limits i.e. $\lim_{x\to a}(fg)(x) = lm$ k_1
   (e) The limit of a quotient is equal to the quotient of the limits provided the limit of the denominator is not zero i.e., $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{l}{m}$, provided that the $\lim_{x\to a} g(x) = m \neq 0$. k_1
   (f) Let f be defined on D and let $f(x) \geq 0 \forall x \in D$. If $\lim_{x\to a} f(x) \geq 0$. k_1
   (g) Let f and g be defined on D and $let\ f(x) \geq g(x) \forall x \in D$. Then $\lim_{x\to a} f(x) \geq \lim_{x\to a} f(x)$, Provided these limits exist. k_1
   (h) (Sand witch Theorem) Let f, g and h be defined on D and let $f(x) \geq g(x) \geq h(x) \forall x$. Let $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Then $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. k_1
   (i) If $\lim_{x\to a} f(x) = l$, then $\lim_{x\to a}|f(x)| = |l|$ k_1
   (j) If a function is continuous in $[a, b]$ then it is bounded and it attains its bounds at least once in $[a, b]$. k_1
   (k) A continuous function on the close interval $[a, b]$ is always bounded. k_1
   (l) A function continuous on $[a, b]$ is also uniformly continuous on the interval. k_1
   (m) Uniformly continuity always implies continuity, but converse need not be true.
   (a) It can also be shown that the limit of the constant multiples, powers and roots. k_1
2. Define the limit (i) Geometrically (ii) by Cauchy's definition (iii) Alternative approach or explain Right hand limit and Left-hand limit on a function. k_2
3. A function is defined by $f(x) = \begin{cases} \frac{x^2}{a} - a & 0 < x < a \\ a - \frac{x^3}{a^2} & x > a, \end{cases}$, and $f(0) = 0$ then show that it is continuous at $x = a$. k_2
4. Discuss the continuity of function $f(x)$. if $f(x) = \begin{cases} 2x & 0 \leq x < 1 \\ 3 & x = 1 \\ 4x & 1 < x \leq 2 \end{cases}$ k_2
5. Discuss the continuity of function $f(x) = \begin{cases} \frac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ k_2
6. What condition can be imposed on p for $f(x) = \begin{cases} x^p \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ to be continuous on $x = 0$. What happens if $\cos\frac{1}{x}$ is replaced by $\sin\frac{1}{x}$ in above function. k_3

7. Discuss the continuity of the function $f(x)$ where
$f(x) = \begin{cases} -x^2 & x \le 1 \\ 4x^2 - 3x & 1 < x < 2 \end{cases}$, $g(x) = \begin{cases} 5x - 4 & 0 < x < 2 \\ 3x + 4 & x \ge 2 \end{cases}$ k_3
8. If a function $f(x)$ is continuous in a interval I. prove t hat $|f(x)|$ is also continuous in I. Is the converse of the theorem true? If not, give an example in support of your answer. k_3
9. Discuss the continuity of the function: k_3
$$f(x) = \begin{cases} \frac{1}{5}(2x^2 + 3) & x \le 1 \\ 6 - 5x & 1 < x < 3 \\ x - 3 & 3 \le x \end{cases}$$
10. Using $(\in, \delta)$ definition show that $\lim_{x \to 0} x \sin\frac{1}{x} = 0$. k_2
11. What is Uniform continuity? k_1
12. Give an example to show that a function continuous in an open interval may fall to be uniformly continuous in the interval. k_2
13. Show by $(\in, \delta)$ method that the function f, defined on $\mathbb{R}\sim\{0\}$ by $f(x) = \sin\left(\frac{1}{x}\right)$ whenever $x \neq 0$, does not tend to 0 as x tends to 0. **Hint: use Uniform continuity.** k_3
14. Evaluate $\lim_{x \to 0} \frac{(1+x)^{\frac{1}{3}} - (1-x)^{\frac{1}{3}}}{x}$ k_3
15. Find the right hand and the left hand limits in the following cases and discuss the existence of the limit in each case: k_3
(i) $\lim_{x \to 0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$ (ii) $\lim_{x \to 0} f(x)$, where $f(x)$ is defined as: $f(x) = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$
16. Let $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ a - x & \text{if } x \text{ is irrational} \end{cases}$ Show that $\lim_{x \to 0} f(x)$ exists only at 0. k_3
17. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exists, can $\lim_{x \to a}[f(x) + g(x)]$ or $\lim_{x \to a}[f(x)g(x)]$ exists? k_3
18. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a}[f(x)g(x)]$ both exists, $\lim_{x \to a} g(x)$ exist? k_3
19. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a}[f(x)g(x)]$ both exists, must $\lim_{x \to a} g(x)$ exist? k_1
20. Show that $\lim_{x \to 0} f(x) = \lim_{x \to a} f(x - a)$ k_2
21. If $x \to 0$, then does the limit of the following function f exist or not? k_2
$$f(x) = \begin{cases} x & x < 0 \\ 1 & x = 0 \\ x^2 & x > 0 \end{cases}$$
22. What is continuity? Define Cauchy's definition of continuity. k_2
23. Discuss Discontinuity and its all types. k_2
24. If $f(x) = x^p$, for what values of p the function $f(x)$ is continuous and differentiable at $x = 0$. k_3
25. Is the following function continuous or not? $f(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ k_2
26. Discuss the continuity of the following function if k_3

$$f(x)=\begin{cases}0 & x=0\\ \frac{1}{2}-x & 0<x<\frac{1}{2}\\ \frac{1}{2} & x=\frac{1}{2}\\ \frac{3}{2}-x & \frac{1}{2}<x<1\\ 1 & x=1\end{cases}$$ also draw its graph.

27. Show that the function $f(x)$ defined on $\mathbb{R}$ by k_3
$f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ -1 & \text{if } x \text{ is irrational}\end{cases}$ is discontinuous at every point of $\mathbb{R}$.
28. Define Limit (L) and Continuity to a point (x_0, y_0) in two dimensions for f(x,y)
29. For the limit to exist, the same limiting value must be obtained whatever direction means what?
30. Explain a crucial difference between the limit of a function of two and a single variable.
31. Find $\lim_{(x,y)\to(x0,y0)} \frac{4xy^2}{x^2+y^2}$ if it exists. Ans: Limit exists and 0
32. Show that the function $f\ (x,y)\ =\ \dfrac{2x^2y}{x^4+y^2}$ has no limit as (x y) approaches (0,0).
33. At what points (x y) in the plane are the following functions continuous?
    (i) $f(x,\ y)=sin(x\ +y)$ Ans : **All (x y).**
    (ii) $g(x,\ y)=sin\dfrac{1}{xy}$ Ans: **All (x y) except where x = 0 or y = 0.**
34. Find the limits of the following.
    (i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2-y^2+5}{x^2+y^2+2}$ **Ans:** $\dfrac{5}{2}$
    (ii) $\lim\limits_{(x,y)\to(0,0)} \cos\dfrac{x^2+y^3}{x+y+1}$ **Ans: 0**
35. Find the limits by rewriting the fractions first.
    (i) $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^2-2xy+y^2}{x-y}$ $x\neq y$ **Ans: 0**
    (ii) $\lim\limits_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4}$ $2x-y\neq 4$ **Ans:** $\dfrac{1}{4}$
36. If $f(x_0,y_0)=3$, what can you say about $\lim\limits_{(x,y)\to(x_0,y_0)}=f(x,y)$, if f is continuous at (x_0,y_0) ? If f is not continuous at (x_0,y_0)? Give reasons for your answer. **Ans: If f is not continuous at (x_0,y_0), the limit could have any value different from 3, and need not even exist**
37. Does knowing that $\left|sin\left(\dfrac{1}{x}\right)\right|\leq 1$ tell you anything about $\lim\limits_{(x,y)\to(0,0)}=y\sin\dfrac{1}{x}$? Give reasons for your answer. **Ans:** $y\sin\dfrac{1}{x}\to 0$ **by Sandwich Theorem**
38. De ne f(0,0) in a way that extends $f(x,y)=xy\dfrac{x^2-y^2}{x^2+y^2}$ to be continuous at the origin.
    Ans: $\lim\limits_{(x,y)\to(0,0)}\left(xy\dfrac{x^2-y^2}{x^2+y^2}\right)=0$, **by Sandwich Theorem**