B. Tech Sem-I, Tutorial-3

Topics: Successive Differentiation, Leibnitz's rule and its applications

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1. Knowledge / Remembering

State **Leibnitz's** theorem to get n^{th} derivative of the product of two functions.

2. Comprehension / Understanding

A. Explain the meaning of successive differentiation with a simple example.

B. Illustrate Leibnitz's theorem by differentiating (x^3 e^x) twice.

3. (Application) Find the n^{th} derivative of (a) $x^3 \log x$ Ans: (a) $(-1)^n(n-4)!\,6\,x^{-n+3}$;

(b) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, **Ans:** $(-1)^{n-1}(n-1)!\,sin^n\theta.\sin n\theta$;

(c) $\cos^2 x \sin^3 x$ **Ans:** $\frac{1}{16}\{2\sin\left(x+n\frac{\pi}{2}\right)+3^n\sin\left(3x+n\frac{\pi}{2}\right)-5^n\sin\left(5x+n\frac{\pi}{2}\right)\}$;

(d) $e^x \sin 4x \cos 6x$, **Ans:** $e^x\frac{(101)^{\frac{n}{2}}}{2}sin(10x+n\,tan^{-1}10)-e^x\frac{(5)^{\frac{n}{2}}}{2}sin(2x+n\tan^{-1}10)$;

(e) $\frac{x+2}{x+1}+\log\left(\frac{x+2}{x+1}\right)$ **Ans:** $(-1)^n\left\{\frac{n!}{(1+x)^{n+1}}-\frac{(n-1)!}{(2+x)^n}+\frac{(n-1)!}{(1+x)^n}\right\}$

(f) $\frac{x^2}{2x^2+7x+6}$ **Ans:** $(-1)^n n!\left\{\frac{2^{n-1}.9}{(3+2x)^{n+1}}-\frac{4}{(2+x)^{n+1}}\right\}$

4. (Analysis) Compare two methods of finding the nth derivative of x^{2n} prove that

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2.2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2}+\ldots=\frac{(2n)!}{(n!)^2}$$

5. (Synthesis / Creating) Combine results of different nth derivative problems to establish a generalized pattern for polynomial–exponential products.

6. (Evaluation)

i. Prove that if $y=(x^2-1)^n$, then successive derivatives satisfy a recurrence relation $(1-x^2)y_{n+2}-2x\,y_{n+1}+n(n+1)y_n=0$.

ii. If $y=\sin(m\sin^{-1}x)$, then successive derivatives satisfy a recurrence relation $(1-x^2)y_{n+2}-(2n+1)x\,y_{n+1}+(m^2-n^2)y_n=0$ hence evaluate $(y_n)_0$.

iii. If $\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^n$, then successive derivatives satisfy a recurrence relation $x^2y_{n+2}+(2n+1)x\,y_{n+1}+2n^2y_n=0$.

iv. If $y=[x-\sqrt{(x^2-1)}]^m$ then successive derivatives satisfy a recurrence relation $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$.

v. If $y=e^{a\,si^{-1}x}$, then successive derivatives satisfy a recurrence relation $(1-x^2)y_{n+2}-(2n+1)x\,y_{n+1}-(n^2+a^2)y_n=0$ and hence evaluate $(y_n)_0$.

vi. If $y=\sin mx+\cos mx$ then successive derivatives satisfy a recurrence relation $y_n=m^n\sqrt{\{1+(-1)^n\sin 2mx\}}$

vii. If $y=[x+\sqrt{(x^2+1)}]^m$, then successive derivatives satisfy a recurrence relation $(1+x^2)y_2+xy_1-m^2y=0$ and hence evaluate $(y_n)_0$.

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