B.Tech Semester - II BAS 203/(OLD)KAS 203T

Tutorial Sheet

Engineering Mathematics-II

Unit-I (Matrices)

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Section A (Rank)

1. Each entry of a matrix is unity. Show that its rank is one.
2. If A is a non-zero column vector $(n \times 1)$ then rank of AA^T *is* (A)2, (B)n, (C)0, (d*)1 .
3. If $\rho(A_{3\times3})=2$ and $\rho(B_{3\times3})=3$ then $\rho(AB)$= (a) 5, (b)3,(c*)2 ,(d) 1
4. If A_{5x7} has all entries -1 then rank of A is (a) 7; (b*) 1; (c) 5; (d) 0.
5. If rank of $\begin{bmatrix} K & -1 & 0 \\ 0 & K & -1 \\ -1 & 0 & K \end{bmatrix}$ is 2, then K is (a) Arbitrary (b*) 1, (c) 2, (d)3.
6. If A= $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 3 & 10 \end{bmatrix}$, B= $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, Find $\rho(A)$, $\rho(B)$, $\rho(A+B)$, $\rho(AB)$ and $\rho(BA)$.

   Ans: $\rho(A)=2$, $\rho(B)=1$, $\rho(AB)=1$and , $\rho(BA)=1$ Hint: $\rho\ (Product) \le \rho(Either)$

7. Determine the value of b if the rank of A *is* 3, where $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$

   {Hint: After using Row transformations

   $\begin{matrix} R_4 \leftrightarrow R_3 \\ \sim \\ \mathrm{R_4 - 3R_2} \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & b+6 & 0 \\ b-2 & 0 & 0 & -2 \end{bmatrix}$ Cases: (1) If b=2, Then $|A| = 0$, $\rho\ (A) = 3$.

   (2) If $b = -6$ Then, number of non-zero rows is 3. Therefore $\rho\ (A) = 3$.

8. Reducing A= $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ to its Normal form, find the rank of the matrix? Ans: 3.
9. Find rank of (i) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix}$ Ans: 3. (ii) A=$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ Ans: 2 (iii) B= $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ Ans: 2

   (iv) A=$\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -3/2 \end{bmatrix}$ **Ans:** $\rho(A)=1$ **(v)** $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ **Ans: 2**

   (vi) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$ **Ans:** $\rho = 4$. (vii) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$ ***Ans***. 3

   (viii) $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & 15 \end{bmatrix}$ ***Ans***. 2 (ix) $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ ***Ans***. 3

Section B (Symmetric, Skew-symmetric, Orthogonal, Hermitian and Skew Hermitian Matrices)

1. Define symmetric, skew-symmetric, and orthogonal matrices with one example.
2. Let A be a square matrix over C. Show that $A+A^{\theta}$, AA^{θ}, and $A^{\theta}A$ are Hermitian matrices and $A-A^{\theta}$ a skew-Hermitian matrix.

3. If A=$\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 4\\ 1 & 4 & 3\end{bmatrix}$ then |AdjA| is (a) -4, (b*)4, (c)-2 ,(d) 2

4. If A and B are square matrices of the same order, which of the following is true
(a)$(A+B)^2 = A^2 + 2AB + B^2$,$(b)(A+B)(A-B) = A^2 - B^2$,
$(c)\ (A-B)\ (A+B)\ = A^2 - B^2$,$(d*)\ (A+B)(A-B) + \ (A-B)\ (A+B)\ = 2A^2 - 2B^2$

5. Show that for any square matrix A with real number entries, A + A′ is a symmetric matrix and A – A′ is a skew-symmetric matrix.

6. If A be a square matrix such that A,A^2, $A^3$are non-zero matrices but A^4 is a zero matrix.Then $(I - A)^{-1}$ is (a) A+ $A^2 + A^3$(b)I+A+ A^2 (c) I+$A^2 + A^3$, (D*) I+A+ $A^2 + A^3$

7. If A=$\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$, show that $A^n = \begin{bmatrix}\cos n\alpha & \sin n\alpha\\ -\sin n\alpha & \cos n\alpha\end{bmatrix}$, where n∈$\mathbb{Z}^+$

8. Prove (i) $\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$ is a nilpotent matrix of index 3; (ii). $\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & -2 & 1 & 0\\ 1 & -3 & 3 & -1\end{bmatrix}$ is an involutory matrix (iii) $\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$ is an orthogonal matrix.; (iv) $\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$ is a unitary matrix.

9. Express $\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

10. Check A=$\begin{bmatrix}1 & -1 & 5\\ -1 & 2 & 1\\ 5 & 1 & 3\end{bmatrix}$ B=$\begin{bmatrix}1 & i & 3i\\ -i & 0 & 2-i\\ -3i & 2+i & -1\end{bmatrix}$ and C=$\begin{bmatrix}1 & i & 3i\\ -i & 0 & 2-i\\ -3i & 2+i & -1\end{bmatrix}$ for symmetric matrix, Hermitian.

11. If k is complex number and A be a Hermitian matrix. Will kA be Hermitian?

12. If matrix A=$\begin{bmatrix}i & 1+i & 2-3i\\ -1+i & 2i & 1\\ -2-3i & -1 & 0\end{bmatrix}$ then prove that the conjugate of A is skew Hermitian

13. Check whether matrix $\begin{bmatrix}3 & 1+i & 2-3i\\ -1+i & 2i & 1\\ -2-3i & -1 & 0\end{bmatrix}$ is a skew-Hermitian matrix.

Section C (Characteristic equation, Cayley-Hamilton theorem, Diagonalization of a Matrix)

1. Prove that all the eigen values of a Hermitian matrix are real.
2. Prove that the eigen values of a triangular matrix are just diagonal elements of the matrix.
3. Find all the eigen values and eigenvectors of the matrices
(i) A=$\begin{bmatrix}3 & 1 & 4\\ 0 & 2 & 6\\ 0 & 0 & 5\end{bmatrix}$Ans: 2,3,5 ;$\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}$, $\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$, $\begin{bmatrix}3\\ 2\\ 1\end{bmatrix}$ (ii)B=$\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ Ans: -3,-3,5 ;$\begin{bmatrix}c\\ d\\ \frac{c+2d}{3}\end{bmatrix}$, $\begin{bmatrix}-c\\ -2c\\ c\end{bmatrix}$,

4. Verify Cayley-Hamilton theorem for the following matrices and hence find A^{-1}
(i) A=$\begin{bmatrix}4 & 3 & 1\\ 2 & 1 & -2\\ 1 & 2 & 1\end{bmatrix}$ $A^{-1}=\frac{1}{11}\begin{bmatrix}5 & -1 & -7\\ -4 & 3 & 10\\ 3 & -5 & -2\end{bmatrix}$, (ii)A=$\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ $A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$

5. A square matrix is defined by A=$\begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$, Is it diagonalizable? If yes, reduce it into diagonal form. Ans: Diag$(1, -1, 3)$

6. Prove that if λ is an eigen value of a matrix A, then kλ, $\boldsymbol{\lambda \pm k}$, $\lambda^n, \frac{1}{\lambda}, \frac{|A|}{\lambda}$ are the eigen values of kA, A$\pm kI, A^n, A^{-1}$, Adj A respectively.

Section D (System of linear equations)

1. If the system of equations $3x - y + \lambda z = 1, 2x + y + z = 2, x + 2y - \lambda z = -1$ has a unique solution if λ has (a) any value, (b) $\frac{-7}{2}$, (c*) $\neq \frac{-7}{2}$, (d) $\neq \frac{7}{2}$.

2. Check the consistency for the following linear systems. If system is consistence find the

$x_1 + x_2 + 2x_3 + x_4 = 5$ $2x_1 + 3x_2 - x_3 - 2x_4 = 2$ $4x_1 + 5x_2 + 3x_3 = 7,$ **Ans: No solution**	$2x + 6y + 11 = 0$ $6x + 20y - 6z = -3$ $6y - 18z + 1 = 0$ **Ans: No solution**	$x + 2y - z = 3$ $3x - y + 2z = 1$ $2x - 2y + 3z = 2$ $x - y + z = -1$ **Ans: $x = -1, y = 4, z = 4$**

3. If the system of equations $x + 4ay + az = 0, x + 3by + bz = 0, x + 2cy + cz = 0$ has a non-trivial solution, then a, b, c, are in (a) A.P, (b)G.P, (c*) H.P., (d) None of these.
4. For what values of λ and μ so that the equations $x + y + z = 6; x + 2y + 3z = 10; x + 2y + \lambda z = \mu$ have (i) No solution; (ii) a unique solution and (iii) an infinite number of solutions ANS**:** (i) $\lambda = 3$ no solution; (ii)$\lambda \neq 3$, $\mu \neq 10$, unique solution; (iii)$\lambda = 3$, $\mu = 10$, Infinite many solutions.
5. Determine b such that the system $2x + y + 2z = 0; x + y + 3z = 0; 4x + 3y + bz = 0$ has (i) trivial solution(ii) non- trivial solution**.**
6. Find the values of a , b and c for which the system has (i) No solution (ii) unique solution (iii) Infinitely many solutions for

$-2x + y + z = a;$ $x - 2y + z = b;$ $x + y - 2z = c$ **Ans**: $(i) If\ a + b + c \neq 0$ *No solution* $(ii) If\ a + b + c = 0$,Infinite solutions	$x + ay + z = 3$ $x + 2y + 2z = b$ $x + 5y + 3z = 9$ **Ans:** (i) $a = -1, b \neq 6$ *no solution* (ii)$a \neq -1, b$ *is arbitrary, unique solution* (iii)$a = -1, b = 6$ *Infinite many* solutions.

Unit–II

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Topic:

On Ordinary Differential Equation of Higher Order:

Linear differential equation of nth order with constant coefficients, Simultaneous linear differential equations, Second order linear differential equations with variable coefficients, Solution by changing independent variable, Reduction of order, Normal form, Method of variation of parameters, Cauchy-Euler equation, Series solutions (Frobenius Method).

Section-A (Linear differential equation with constant coefficient)

Solve the following differential equation

Q1. $\left(D^3+6D^2+11D+6\right)y=0$ $\left[Ans.y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}\right]$

Q.2 $\left(D^4-D^3-9D^2-11D-4\right)y=0$ $\left[Ans.y=\left(c_1+c_2x+c_3x^2\right)e^{-x}+c_4e^{4x}\right]$

Q.3 $\left(D^3+D^2+D+1\right)y=\sin x$ $\left[Ans.y=c_1e^{-x}+c_2\cos x+c_3\sin x+\frac{1}{15}\left(2\cos 2x-\sin 2x\right)\right]$

Q.4 $\left(D^3-3D^2+4D-2\right)y=e^x+\cos x$

$\left[Ans.y=c_1e^x+e^x\left(c_2\cos x+c_3\sin x\right)+xe^x+\frac{1}{10}\left(\cos x+3\sin x\right)\right]$

Q.5 $\left(D^3+3D^2+2D\right)y=x^2$ $\left[Ans.y=c_1+c_2e^{-x}+c_3e^{-2x}+\frac{x}{12}\left(2x^2-9x+21\right)\right]$

6. Solve $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$. Ans: $y=e^{-2x}\left(A\text{Cos}\,x+B\text{Sin}\,x\right)$.

7. Solve $\left(D^2+6D+9\right)y=5e^{2x}+e^{-3x}$. Ans: $y=\left(C_1+C_2x\right)e^{-3x}+\frac{1}{5}e^{2x}+\frac{x^2}{2}e^{-3x}$.

8. Find P.I. to the differential Equation $\left(D^2+4\right)y=x$ .Ans: $P.I.=\frac{x}{4}$.

9. Solve the differential Equation $\left(D^2-3D+2\right)y=6e^{3x}+\text{Sin}\,2x$

Ans: $y=C_1e^x+C_2e^{2x}+3e^{3x}+\frac{1}{20}\left(3\text{Cos}\,2x-\text{Sin}\,2x\right)$

10. Solve the differential Equation $\left(D^2-4D+1\right)y=e^{2x}\,\text{Sin}\,2x$

Ans: $Y = \left(C_1 \operatorname{Cosh}\sqrt{3}x + C_2 \operatorname{Sinh}\sqrt{3}x\right)e^{2x} - \frac{1}{7}e^{2x}\operatorname{Sin}2x$

11. Solve the differential Equation $\left(D^2 - 3D + 2\right)y = e^{5x}$

Ans: $y = C_1 e^x + C_2 e^{2x} + \frac{1}{12}e^{5x}$.

Section-B (Homogeneous Linear Differential Equation with Variable Coefficient (Cauchy-Euler Equation)):

Q.1 $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \left(x + \log x\right)$

$$\left[Ans.y = c_1\frac{1}{x} + c_2\sqrt{x}\cos\left(c_3 + \frac{\sqrt{3}}{2}\log x\right) + \frac{x}{2} + \log x\right]$$

Q.2 $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 20y = \left(x+1\right)^2$

$$\left[Ans.y = c_1x^{-5} + c_2x^{-4} - \frac{1}{14}x^2 - \frac{1}{9}x - \frac{1}{20}\right]$$

Q.3 $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x \qquad \left[Ans.y = c_1x^3 + c_2x^2 + \frac{x}{2}\right]$

Q.4 $x^4\frac{d^3y}{dx^3} - 2x^3\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 1 \qquad \left[Ans.y = \left(c_1 + c_2\log x\right)x + \frac{c_3}{x} + \frac{\log x}{4x}\right]$

5. Solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2$. **Ans:** $y = c_1x^2 + c_2x^3 - x^2\log x$

6. Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x.$

Ans: $y = \frac{c_1}{x} + x\left[c_2\cos(\log x + c_3\sin(\log x)\right] + 5x + 10\frac{\log x}{x}$

7. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x.$ **Ans:** $y = \left(c_1 + c_2\log x\right)x + 2\log x + 4$

8. Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$. **Ans:** $y = \left(c_1 + c_2\log x\right)x^2 + x^2\left(\log x\right)^2$.

9. Solve $x^3\frac{d^3y}{dx^3} - 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$

Ans: $y = \sqrt{x}\left[c_1\cos\left(In\,x\sqrt{\frac{3}{2}}\right) + c_1\sin\left(In\,x\sqrt{\frac{3}{2}}\right)\right] + x^2$.

Section-C (Legendre 's Linear Differential Equation)

1. Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\left(\log(1+x)\right)$

Ans: $y=(c_1\cos t+c_2\sin t)+2t\sin t, t=\log(1+x).$

2. Solve $(2x+3)^2\frac{d^2y}{dx^2}-(2x+3)\frac{dy}{dx}-12y=6x$

Ans: $y(x)=c_1(2x+3)^{m_1}+c_2(2x+3)^{m_2}-\frac{3}{14}(2x+3)+\frac{3}{4}$

3. Solve $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1$

Ans: $y(x)=c_1e^{2t}+c_2e^{-2t}+\frac{te^{2t}+1}{108}.$

4. Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=2\sin\left(\log(1+x)\right)$

Ans: $y=c_1\cos(\log(1+x))+c_2\sin(\log(1+x)-\log(1+x)\cos(\log(1+x).$

5. Solve $(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2$

Ans: $y=(1+2x)^2\left\{\left[\log(1+2x)\right]^2+c_1\log(1+2x)+c_2\right\}.$

Section-D (Solution of Second order differential equation equation whose one solution is known):

1.Solve $x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}-y=e^x$. **Ans:** $y=(\log x+c_1\int\frac{e^{-x}}{x}dx+c_2)e^{x^2}$.

2.Solve $(3-x)\frac{d^2y}{dx^2}-(9-4x)\frac{dy}{dx}+(6-3x)y=0.$

Ans: $y=e^{3x}\frac{c_1}{8}(4x^3-42x^2+150x-183)+c_2e^x).$

3. Solve $x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$. **Ans :** $y=x\left(-\frac{x}{2}+\frac{C_1}{2}e^{2x}+C_2\right)$

4. Solve $(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=(x+1)e^x.$

Ans : $y = e^{2x}\left[-e^{-x} + \frac{Ce^{-2x}}{4}(2x+5) + C_1\right]$.

5. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$. **Ans :** $y = e^x\left[-\sin x + c_1 x + c_2\right]$

Section-E (Normal form (Removal of First derivative)):

1.Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$. **Ans:** $y = (c_1e^{-x} + c_2e^{x} - 1)e^{x^2}$.

2.Solve $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + y = 0$. **Ans:** $y = (c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x)\sec x$.

3.Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = 0$. **Ans:** $y = (c_1\cos x + c_2\sin x)\ x$.

4. Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4x^2y = e^{x^2}$. **Ans:** $y = \left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{1}{2}\right)e^{x^2}$.

5. Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\ \mathrm{Sin}\,2x$.**Ans :** $y = e^{x^2}(c_1\cos x + c_2\sin x + \sin 2x)$.

Section-F (Simultaneous Differential Equation):

1. Solve $\frac{dx}{dt} + 5x - 2y = t, \frac{dy}{dt} + 2x + y = 0.$

Ans: $x(t) = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{27}(1+3t), .\ y(t) = -\frac{2}{27}(2+3t)e^{-3t} + \frac{2}{27}(2-3t)$

2. Solve $\frac{dx}{dt} + 2x + 3y = 0, \frac{dy}{dt} + 3x + 2y = 2e^{2t}.$

Ans: $x(t) = e^t + e^{-t}, y(t) = e^{-t} - e^t + \sin t$.

3. Solve $\frac{dx}{dt} + 2y = e^t, \frac{dy}{dt} - 2x = e^{-t}$

Ans: $x(t) = -c_1\sin 2t + c_2\cos 2t + \frac{e^t}{5} - \frac{2}{5}e^{-t}, y(t) = (c_1\cos 2t + c_2\sin 2t) + \frac{2e^t}{5} - \frac{e^{-t}}{5}$.

4. Solve $\frac{dx}{dt} + x + 2\frac{dy}{dt} + y = e^t, \frac{dx}{dt} - x + \frac{dy}{dt} + y = 1.$

Ans: $x(t) = c_1e^t + c_2e^{-2t} + 2e^{-t}, y(t) = 3c_1e^t + 2c_2e^{-2t} + 3e^{-t}$

5. Solve $\frac{dx}{dt} - 7x + y = 0, \frac{dy}{dt} - 2x - 5y = 0$

Ans: $x(t)=e^{6t}\left(c_1\cos t+c_2\sin t\right),\ \ y(t)=e^{6t}\left[\left(c_1-c_2\right)\cos t+\left(c_1+c_2\right)\sin t\right]$

Section-G (Method of Variation of Para Meters):

1. By method of variation of parameters solve the following Differential equation y'' + y = Sec x.

Ans : y = $u\cos x+v\sin x=(\log\cos x+c_1)\cos x+(x+c_2)\sin x.$

2. Solve the following differential equation by method of variation of parameters $\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}$

Ans : $y=C_1e^x+C_2e^{-x}-1+e^x\log\left(e^{-x}+1\right)-e^{-x}\log(e^x+1).$

3. By method of variation of parameters solve the following Differential equation y'' - y = $e^{-2x}\sin e^{-x}$.

Ans : y = $(c_1e^x+c_2e^{-x})-(\sin e^{-x}+e^x\cos e^{-x}).$

4. Solve the following differential equation by method of variation of parameters $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=e^x\tan x$.

Ans : $y=e^x(c_1\cos x+c_2\sin x)-e^x\cos x\log\left(\text{Sec}\,x+\tan x\right).$

5. Solve the following differential equation by method of variation of parameters $\frac{d^2y}{dx^2}+y=x\sin x$.

Ans : $y=(c_1\cos x+c_2\sin x)+\frac{1}{8}\cos x+\frac{x}{4}\sin x-\frac{x^2}{4}\cos x.$

Section-H (Solution of Second order differential equation by changing independent variable):

1.Solve $x\frac{d^2y}{dx^2}-\frac{dy}{dx}+4x^3y=x^5$. **Ans:** $y=c_1\cos x^2+c_2\sin x^2+\frac{1}{4}x^2$.

2.Solve $\frac{d^2y}{dx^2}+\tan x\ \frac{dy}{dx}+\cos^2 x\,y=0$. **Ans:** $y=c_1\sin(\sin x+c_2)$.

3.Solve $\frac{d^2y}{dx^2}+(8e^{2x}+2)\frac{dy}{dx}+4e^{4x}\,y=e^{6x}$. **Ans:** $y=c_1e^{(2+\sqrt{3})e^{2x}}+c_2e^{(2-\sqrt{3})e^{2x}}+\frac{1}{8}(e^{2x}+4).$

4.Solve $x^6\frac{d^2y}{dx^2}+3x^5\frac{dy}{dx}+a^2y=\frac{1}{x^2}$. **Ans:** $y=c_1\cos\frac{a}{2x^2}-c_2\sin\frac{a}{2x^2}+\frac{1}{a^2x^2}$.

5. Solve $\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}+4x^2y=x^4$.**Ans:** $y=c_1\cos(x^2)+c_2\sin(x^2)+\frac{x^2}{4}$.

6.Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log(1+x)$

Ans: $y=c_1\cos\log(1+x)+c_2\sin\log(1+x)+2\log(1+x)\sin\log(1+x)$.

7._Solve $\cos x\frac{d^2y}{dx^2}+\sin x\frac{dy}{dx}-2y\cos^3x=2\cos^5x.$

Ans: $y=c_1e^{\sqrt{2}\sin x}+c_2e^{-\sqrt{2}\sin x}+(\sin x)^2$

8. Solve $(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}+4y=0.$

Ans: $y=c_1\cos2(\tan^{-1}x)+c_2\sin2(\tan^{-1}x)$

Section-I (Picard' s Method)

Q.1 Using Picard's process of successive approximations, obtain a solution up to the fifth approximation of the equation dy/dx = y + x, such that y = 1 when x = 0.

Ans: $y_5=1+x+x^2+\frac{x^3}{3}+\frac{x^4}{12}+\frac{x^5}{60}+\frac{x^6}{720}$

Q.2 Find the value of y for x = 0.1 by Picard's method, given that

$$\frac{dy}{dx}=\frac{y-x}{y+x},\ y(0)=1.$$

Ans: y (0.1) = 1 – (0.1) + 2 log 1.1 = 0.9828.

Q.3 Using Picard's method, solve $\frac{dy}{dx}=2y$ with $x_0=0, y_0=1.$

Ans: $y=\sum_{i=1}^{n}\frac{(2x)^i}{i!}$

Q.4 Employ Picard 's method to obtain solution of the differential equation correct up to third approximation $\frac{dy}{dx}=x+y^2$ for $x=0.3$, given that $y=0$ when $x=0$.

Ans: $y=\frac{x^2}{2}+\frac{x^5}{20}+\frac{x^8}{160}+\frac{x^{11}}{4400}\simeq0.0451 \quad at\ \ 0.3.$

Q.5 Obtain Picard's second approximate solution of the initial value problem,

$$y' = 2x(1+y), y(0) = 0 .$$

Ans: $y = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots\ldots\ldots + \frac{x^{2n}}{n!}$

UNIT-III
Section-A (Fourier Series)

Prepared by Prof Raj Kumar

Preamble: **(i).** $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(ii) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

(iii) $\int u\,v\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...,$ u, v are functions of x and dashes denote differentiation and suffixes denote integration with respect to x

(iv) Fourier series for $f(x)$ in $c \le x \le c+2\pi$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$

where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$

, $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$

(v) Fourier series for $f(x)$ in $c \le x \le c+2l$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l},$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$

, $b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$

(vi) The half range cosine series in $0 \le x \le l$is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$, $a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

, and half range sine series in $0 \le x \le l$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ where $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

Q.1 Express $f(x) = \frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$. Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$ **Ans:**

$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

Q.2 Find the Fourier series to represent the function $f(x) = |\sin x|$, $-\pi < x < \pi$.

Ans:

$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + ..., \frac{\cos 2nx}{4n^2-1} + ...\right)$

Q.3 Express $f(x) = |x|$, $-\pi < x < \pi$,as a Fourier series. Hence show that

$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$. **Ans:**

$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{cos3x}{3^2} + \frac{cos5x}{5^2} + \ldots\right)$

Q.4 Find the Fourier series for function Hence show that **Ans:**

$f(x) = \frac{\pi^2}{3} + 4\left(\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots\right) - 2\left(\frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + ..\right)$

Q.5 Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$.

Ans:

$x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \ldots\right)$

Q.6 Find the Fourier series to represent the function $f(x) = \{$. Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots$

Ans: $f(x) = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$

Q.7 Find the Fourier series of $f(x) = \{$ which is assumed to be periodic with period 2π.

Ans: $f(x) = \frac{-\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\right)$

Q.8 Obtain Fourier series for function

Ans: $\frac{\pi}{2} - \frac{4}{\pi} \le \left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right)$

Q.9 Find the Fourier series to represent $f(x) = x^2 - 2,\ -2 \le x \le 2$.

Ans:

$x^2 - 2 = -\frac{2}{3} - \frac{16}{\pi^2}\left(\cos \frac{\pi x}{2} - \frac{1}{4}\cos \pi x + \frac{1}{9}\cos \frac{3\pi x}{2} - \ldots\right)$

Q.10 Obtain the half range sine series for $f(x) = e^x,\ 0 < x < 1$.**Ans:**

$e^x = 2\pi \sum_{n=1}^{\infty} \frac{n[1-e(-1)^n}{1+n^2\pi^2} sin\, n\pi x$

Q.11 Express $\sin x$ as a cosine series in $0 < x < \pi$.

Ans: $\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \ldots\right)$

Q.12 Obtain the half-range sine series for the function $f(x) = x^2$ in the interval$(0, 3)$.

Ans: $f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{3}$, *where* $b_n = \frac{18}{n\pi}(-1)^{n+1} + \frac{36}{n^3\pi^3}[(-1)^n - 1]$

Section-B (Laplace and Inverse Laplace Transform)

Q. 1 Find the Laplace transform of the following functions:

(i) $\sin h^3 3t$ **(ii)** $\sin h\, 3t \cos 7t$ **(iii)** $t^2 e^{-3t} \sin 5t$ **(iv)** $e^{-4t} \sin 3t \sin 5t$

Q. 2 Find the Laplace transform of the functions

(i) $f(t) = |t-1| + |t+1|,\ t \geq 0$ **(ii)** $f(t) = \begin{cases} \sin(t-\pi/3), & t > \pi/3 \\ 0 & , t < \pi/3 \end{cases}$ (iii) $\sin t \cos t \log t\, \delta(t-\pi)$ where, $\delta(t-\pi)$ is Unit Impulse function.

Q.3 Find the Laplace transform of the functions

(i) $f(t) = \dfrac{\cos\sqrt{t}}{\sqrt{t}}$ **(ii)** $f(t) = \dfrac{\sin at}{t}$ where, a is constant. Does the Laplace transform of $f(t) = \dfrac{\cos at}{t}$ exist

Q. 4 By using the Laplace transform evaluate the following integrals

(i) $\int_0^\infty \dfrac{\sin t}{t} dt$ **(ii)** $\int_0^\infty t^3 e^{-t} \sin t\, dt = 0$ **(iii)** $\int_0^\infty \dfrac{e^{-t} \sin^2 t}{t} dt$

Q. 5 Find the Laplace transform of the $erf\sqrt{t}$

Q. 6 Find a function $f(t)$ for each of the given function $F(s)$ given bellow such that $L\{f(t)\} = F(s)$

(i) $\dfrac{1}{\sqrt{s^2+4}}$ **(ii)** $\dfrac{1}{s^{7/2}}$ **(iii)** $\dfrac{5}{s^2} + \left(\dfrac{\sqrt{s}-1}{s}\right)^2 - \dfrac{7}{3s+2}$ **(iv)** $\dfrac{3s+7}{s^2-2s-3}$ **(v)** $\dfrac{32}{(16s^2+1)^2}$ **(vi)** $\dfrac{e^{4-3s}}{(s+4)^{5/2}}$ **(vii)** $\dfrac{s^2}{(s^2+4)^2}$

Q. 7 Find the values of the following by using the convolution theorem

(i) $L^{-1}\left[\dfrac{s}{(s^2+16)^2}\right]$ **(ii)** $L^{-1}\left[\dfrac{1}{(s-2)(s^2+1)}\right]$ **(iii)** $L^{-1}\left[\dfrac{1}{s(s^2+4)^2}\right]$ **(iv)** $L^{-1}\left[\cot^{-1}\dfrac{s+3}{2}\right]$ **(v)** $L^{-1}\left[\log\dfrac{s+a}{s+b}\right]$

Q. 8 Find solution of each of the following initial value problem by using Laplace transform

(a) $y''-3y'+2y = 4t+e^{3t}$ $\quad y(0)=1,\ y'(0)=-1$

(b) $y'''+2y''-y'-2y=0$ $\quad y(0)=1,\ y'(0)=2,\ y''(0)=2$

(c) $t\,y''+(1-2t)y'-2y=0$ $\quad y(0)=1,\ y'(0)=2$

(d) $y''-ty'+y=1$ $\quad y(0)=1,\ y'(0)=2$

(e) $y'+3y = \begin{cases} t, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$ $\quad y(0)=2$

(f) $y''+5y'+6y = 1-u(t-3)-u(t-5)$ $\quad y(0)=0,\ y'(0)=0$

Q. 9 Solve the following simultaneous equations by using Laplace transform

(a) $(D^2-3)x-4y=0,\ x+(D^2+1)y=0$ given that $x = y = \dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} = 2$ at $t=0; D \equiv \dfrac{d}{dt}$

(b) $(D^2-1)x+5Dy = t,\ 2Dx-(D^2-4)y = 2$ given that $x = y = \dfrac{dx}{dt} = \dfrac{dy}{dt} = 0$ at $t=0;\ D \equiv \dfrac{d}{dt}$

Q. 10 Solve the following initial-boundary value problems.

$(a).\ u_x + x u_t = 0,\quad u(x,0)=1,\ u(0,t)=t$

$(b).\ u_{tt} = u_{xx},\ x>0,\ t>0,\ u(x,0)=e^{-x},\ u_t(x,0)=0,\ u(0,t)=0,\ u(x,t)$ *is bounded as* $x \to \infty$

Answers:

1. i. $\dfrac{162}{(s^2-81)(s^2-9)}$, **ii.** $\dfrac{3(s^2-58)}{(s^2+58)^2-36s^2}$, **iii.** $\dfrac{10(3s^2+6s+2)}{(s^2+6s+34)^2}$, **iv.** $\dfrac{30(s+4)}{(s^2+8s+20)(s^2+8s+80)}$.

2. i. $\dfrac{2}{s}(1+\dfrac{e^{-s}}{s})$ **ii.** $\dfrac{e^{-s\pi/3}}{s^2+1}$ **iii.** 0. **3. (i)** $\sqrt{\dfrac{\pi}{s}}e^{-1/4s}$ **(ii)** $\tan^{-1}(1/s)$ **(iii)** no; **4. (i)** $\dfrac{\pi}{2}$ **(ii)** 0 **(iii)** (¼) log 5; **5.** $\dfrac{1}{s\sqrt{s+1}}$

6 i. $(1/2)\sin 2t$; **ii.** $(8/15)\sqrt{t/\pi}$; **iii.** $6t+1-4\sqrt{t/\pi}-(7/3)e^{-27/3}$; **iv.** $4e^{3t}-e^{-t}$;

v. $(t/4)\sin(t/4)$; **vi.** $(4/3\sqrt{\pi})(t-3)^{3/2}\,e^{-4(t-4)}\,U(t-3)$; **vii.** $(1/4)(\sin 2t+2t\cos 2t)$;

7 i. $\dfrac{t\sin 4t}{8}$; **ii.** $(1/5)(e^{2t}-2\sin t-\cos t)$; **iii.** $(1/16)(1-t\sin 2t-\cos 2t)$; **(iv)** $\dfrac{e^{-3t}\sin 2t}{t}$; **(v)** $\dfrac{e^{-bt}-e^{-at}}{t}$;

8 a. $3+2t+(1/2)(e^{3t}-e^{t})-2e^{2t}$; **b.** $(1/3)(5e^{t}+e^{-2t})-e^{-t}$; **c.** e^{2t};

d. $2t+1$; **e.** $[(3t-1+19e^{-3t})u(t)+(1-3t+2e^{-3(t-1)})u(t-1)]/9$; **f.** $(1/6)[1+2e^{-3t}-3e^{-2t}]u(t)$

$-(1/6)[1+2e^{-3(t-3)}-3e^{-2(t-3)}]u(t-3)-(1/6)[1+2e^{-3(t-5)}-3e^{-2(t-5)}]u(t-5)$;

9. a. $x=2t\cosh t, y=(1-t)\sinh t$; **b.** $x=-t+5\sin t-2\sin 2t,\ y=1-2\cos t+\cos 2t$

10. a. $u(x,\ t)=1+[(t-x^2/2)-1]u(t-x^2/2)$; **b.** $u(x,\ t)=e^{-x}\cosh t-\cosh(t-x)u(t-x)$

Unit-IV

Section-A (Introduction of Complex numbers and Analytic functions)

1. Find the locus of z in each of the following relations: $(i)|z-5| = 6$, (ii)$|z+2i| \geq 1$, (iii)$Re(z+2) = -1$, (iv)$|z-i| = |z+i|$, (v)$|z+3| + |z+1| = 4$, (vi)$1 \leq |z-3| \leq 2$, (vii) $|z+3| - |z+1| = 1$.
2. For which complex number following are true, justify in each (i)$z = -z$, (ii)$-z = z^{-1}$, (iii) $z = z^{-1}$, (iv)$z = \bar{z}$
3. Define an analytic function at a point and in a domain.
4. Prove that an analytic function of constant modulus is always constant.
5. Prove that Real and imaginary parts of an analytic function are harmonic.
6. Prove that an analytic function is always continuous but converse need not be true. Give an example.
7. State and prove the necessary and sufficient condition for a function $f(z) = u + iv$ to be analytic.
8. Define an analytic function at a point. Illustrate such a function.
9. If $f(z) = \frac{(\bar{z})^2}{z}, z \neq 0; f(0) = 0$ then $f(z)$ satisfies Cauchy–Riemann equations (CR) at origin.
10. Using Milne–Thomson method construct an analytic function $f(z) = u + iv$ for which $2u + 3v = 13(x^2 - y^2) + 2x + 3y$.
11. State and Prove Cauchy–Riemann equations in polar coordinate system.
12. Let $f(x, y) = \sqrt{(|xy|)}$, then (a) f_x, f_y do not exist at $(0, 0)$; (b) $f_x(0, 0) = 1$; (c) $f_y(0, 0) = 0$; (d) f is differentiable at $(0, 0)$. **Ans**: c
13. If function $f(z) = \left(\frac{\bar{z}}{z}\right)^2$, when $z \neq 0$, $f(z) = 0$ for $z = 0$, Then $f(z)$ (a) satisfies C.R. equations at $z = 0$; (b) is not continuous at $z = 0$; (c) is differentiable at $z = 0$; (d) is analytic at $z = 1$ **Ans**: b
14. Show that $f(z) = \frac{(\bar{z})^2}{z}$, when $z \neq 0$, $f(z) = 0$ for $z = 0$, satisfies C.R. equations at $z = 0$; but is not differentiable at $z = 0$.
15. The harmonic conjugate of $u = x^2 - y^2 + xy$ is (a) $x^2 - y^2 - xy$; (b) $x^2 + y^2 - xy$; (c) $1/2(-x^2 + y^2) + 2xy$. (d) $2(-x^2 + y^2) + 1/2$ Ans: c

16. $f(z) = (|z|)^2$ is (a) continuous everywhere but nowhere differentiable; (b) continuous at $z = 0$ but differentiable everywhere; (c) continuous nowhere; (d) none of these. **Ans**: d

17. Examine the nature of the function $f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$; if $z \neq 0$, otherwise 0, in a region including the origin.

18. Prove that the function $f(z) = u + iv$, where $f(z) = \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$; if $z \neq 0$, otherwise 0 is continuous and that Cauchy–Riemann equations are satisfied at the origin, yet $f'(z)$ does not exists there.

19. If $f(z) = \frac{x^3y(y-ix)}{x^6+y^2}$; if $z \neq 0$ and $f(0) = 0$, show that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

20. If $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$; if $z \neq 0$ and $f(0) = 0$, prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

21. The function $f(z) = z$ is (a) analytic at $z = 0$, (b) differentiable only at $z = 0$;(c) satisfies C.R. equations everywhere; (d) nowhere analytic. **Ans: d**

22. Derive the C.R. equations for an analytic function $f(r, \theta) = u(r, \theta) + i\, v(r, \theta)$ and deduce that $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$.

23. Find the point where the C.R. equations are satisfied for the function $f(z) = xy^2 + ix^2y$. In which region $f'(z)$ exists?

24. $f(z) = (|z|)^2$ is (a) continuous everywhere but nowhere differentiable except at 0; (b) continuous at $z = 0$ but differentiable everywhere (c) continuous nowhere; (d) none of these. **Ans**: a

25. Prove that the function $f(z) = z|z|$ is nowhere analytic.

26. If $f(z)$ is an analytic function such that $Ref'(z) = 3x^2 - 4y - 3y^2$ and $f(1 + i) = 0$ then $f(z)$ is (a) $z^3 + 6 - 2i$, (b) $z^3 + 2iz^2 + 6 - 2i$, (c) $z^3 + 2iz^2 - 2i$, (d) $z^3 + 2z^2 + 6 - 2i$ **Ans**: b
Hint: $u_x = 3x^2 - 4y - 3y^2 = \phi_1(x, y)(say)$, integrating partially w.r.t. y we get $u = x^3 - 4xy - 3xy^2 + g(y)$
Therefore, $u_y = -4x - 6xy + g'(y)$
or $-u_y = v_x = \phi_2(x, y)(say) = 4x + 6xy - g'(y)$
Thus $\phi_1(z, 0) = 3z^2, \phi_2(z, 0) = 4z + g'(0)$
Now, applying Milne Thomson we get
$f(z) = \int(3z^2 + i\, 4z)dz + constt.$
or $f(z) = z^3 + 2i\, z^2 + constt.$ and applying $f(1 + i) = 0$ implies $Constt = 6 - 2i$

27. The orthogonal trajectory of $u = e^x(x\cos y - y\sin y)$ is (a) $e^x(\cos y + x\sin y) + c$; (b) $e^x x\sin y + c$; (d) $e^x(y\cos y + x\sin y) + c$. **Ans**: c

28. Find the locus of points in the plane satisfying the relation $|z + 5|^2 + |z - 5|^2 = 75$. **Ans**: circle

29. The function $f(z) = \bar{z}$ is (a) analytic at $z = 0$, (b) continuous at $z = 0$;(c) differentiable only at $z = 0$; (d) analytic anywhere. **Ans**: b

30. If $f(z) = u + iv$ is an analytic function and $u - v = (x - y)(x^2 + 4xy + y^2)$ then $f(z)$ is (a)$-z^3 + c$, (b) $-iz^2 + ic$, (c)$-iz^3 + \beta$, (d) $z^3 - ic$, **Ans**: c

31. If $f(z) = u + iv$, is an analytic function of z and $u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$; find $f(z)$ if $f(z)$ subject to the condition $f(\pi/2) = 0$. **Ans**: $1/2\{1 - \cot(z/2)\}$

32. If $f(z) = u(r, \theta) + iv(r, \theta)$ is an analytic function and $u = -r^3 \sin 3\theta$ then construct the analytic function $f(z)$.

33. If $f(z) = u + iv$, is analytic function of z and $u + v = \frac{2\sin 2x}{-2\cos 2x + e^{2y} - e^{-2y}}$; find $f(z)$ in terms of z. **Ans**: $\frac{1}{2}(1 + i)\cot z + d$

34. Choose the correct code for matching list A and B.

A (u is given)	B ($f(z) = u + iv$ is an analytic function)
p. $x^3 - 3xy^2 + 3x + 1$	(i). $\sin z + ci$
q. $y^3 - 3x^2y$	(ii).$z^3 + 3z + 1 + ci$
r. $\sin x \cosh y$	(iii).$i(z^3 + c)$

i ii iii
(a) p r q
(b) r p q
(c) p q r **Ans**: b

35. if $\sin(\alpha + i\beta) = x + iy$ prove that (a) $\frac{x^2}{\cosh^2 \beta} + \frac{y^2}{\sinh^2 \beta} = 1$, (b) $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} = 1$.

36. Prove that $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})|f(z)|^2 = 4|f'(z)|^2$ for an analytic function $f(z) = u + iv$.

37. Find the harmonic conjugate of $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2x + 1$ and corresponding analytic function $f(z) = u + iv$. **Ans**: $v = 3x^2y - y^3 + 6xy + 2y + d$, $f(z) = z^3 + 3z^2 + 2z + 1 + id$.

38. Find orthogonal trajectory of $v = e^{2x}(x\cos 2y - y \sin 2y)$ **Ans**: $-e^{2x}(x \sin 2y + y \cos 2y) + d$

Section-B (Complex Integration)

1. Define simply and multiply connected regions? State and prove Cauchys theorem for an analytic function? Is it true for multiply connected regions?
2. State and prove Cauchy integral formula for nth derivative? Is it true for multiply connected regions? If yes, give your explanation with necessary proofs.
3. If $f(z)$ is analytic in a simply connected region D and a, z are two points in D, then $\int_a^z f(z)dz$ is independent of the path in D joining a and z.
4. Evaluate $\int_{(0,3)}^{(2,4)} \bar{z}\, dz$ along (a) parabola $x = 2t, y = t^2 + 3$, (b) a straight line joining $(0, 3)$ and $(2, 4)$. Find whether both values are different, if yes, justify reason why it is so?
5. Evaluate $\int_C \bar{z}\, dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by $(i) z = t^2 + it$; (ii) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$. **Ans:** (i) $10 - \frac{8i}{3}$; $(ii) 10 - 8i$.
6. If $F(a) = \oint_C \frac{z^2+2z-5}{z-a} dz$, where C is the ellipse $(x/2)^2 + (y/3)^2 = 1$. Find the value of $F(4.5)$. **Ans:** Zero
7. Evaluate $\oint_C \frac{1}{z-a} dz$, where C is any simple closed curve and $z = a$ is (i) outside C; (ii) inside C. **Ans:** (i) 0 ; (ii) $2\pi i$.
8. Evaluate $\oint_C \frac{1}{(z-a)^n} dz$; $n \neq 1$, $n \in Z^+$, where C is any simple closed curve and $z = a$ is inside C. **Ans:** Zero
9. Suppose $f(z)$ is analytic inside and on a simple closed curve C and $z = a$ is inside C. Prove that $\oint_C \frac{1}{z-a} dz = \oint_{C_1} \frac{1}{z-a} dz$ where C_1 is a circle center at a and totally contained in simple closed curve C.
10. Evaluate $\oint_C (\bar{z})^2 dz$; where C is the circle $(i)|z - 1| = 1$, $(ii)|z| = 1$ **Ans:** $4\pi i$, 0.
11. Find $\oint_C \bar{z} dz$ around (a) the circle $|z - 2| = 3$, (b) the ellipse $|z - 3| + |z + 3| = 10$, (c) the square with vertices 0, 2, $2i$ and $2 + 2i$. **Ans:** $18\pi i$, $40\pi i$, $8i$
12. Suppose $f(z)$ is integrable along a curve C having finite length l and there exists a positive real number M such that $|f(z)| \leq M$ on C. Prove that $|\int_C f(z)dz| \leq Ml$.
    Remark: This result is helpful to evaluate the upper bound of an integral without evaluating it.
13. Work out the following integrals around the contour prescribed against it
    (i) $\oint_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)(z-2)} dz$; C: $|z| = 3$. **Ans:** $4\pi i$.
    (ii) $\oint_C \frac{1}{z^2+16} dz$; C: $|z| = 6$. **Ans:** 0

(iii) $\oint_C \frac{e^{5z}}{z-2i}dz$; C: $|z-2|+|z+2|=6$. **Ans:** $2\pi i e^{10i}$

(iv) $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz$; C: $|z|=3$. **Ans:** e^2

(v) $\oint_C \frac{\sin \pi z}{z^2-1}dz$;

- C is the rectangle with vertices $2+i$, $2-i$, $-2+i$, $-2-i$;
- C is the rectangle with vertices $-i$, $2-i$, $2+i$, i. **Ans:** 0, 0.

(vi) $\oint_C \frac{e^{\pi iz}}{z^2-4z+5}dz$; C: $|z-1-2i|=2$. **Ans:** $\pi e^{i\pi}$

(vii) $\oint_C \frac{e^z+\cos z}{(z-5)(z+5i)}dz$; where C is the boundary of a triangle with vertices: -1, 1, $-7/2i$.
Ans: $\frac{2\pi i(\pi i-5)}{\pi^2+25}(2\cos \pi i - \sin \pi i)$

(viii) $\oint_C \frac{\sin^6 z}{z-\pi/6}dz$; C: $|z|=2$. **Ans:** $\pi i 2^{-5}$

(ix) Show that $\int_C \frac{e^{z^2}}{z^2(z-1-i)}dz = \pi e^{2i}$; where C consists of $|z|=2$ anticlockwise and $|z|=1$ clockwise.

(x) $\frac{1}{2\pi i}\oint_{C:|z|=4} \frac{e^z}{(z+2)^2}dz$; **Ans:** e^{-2}

14. Evaluate $\oint_C \frac{1}{z^2-1}dz$ around $z=-i+5e^{it}$. **Ans:** 0

15. Evaluate $\oint_C (z^2+3+4/z)dz$, C: $|z|=4$. **Ans:** $8\pi i$

16. Evaluate the upper bound of the integral without evaluating it ?

    (a) $\int_C (z^4+1)dz$, C : Line segment from 0 to $1+i$. **Ans:** $5\sqrt{2}$.

    (b) $\int_C z dz$, C : Line segment from 0 to i. **Ans:** 1.

    (c) $\int_C 2z dz$, C : Line segment from i to $2+i$. **Ans:** $4\sqrt{5}$.

    (d) $\int_C (x^2+iy^2)dz$, C : $z=e^{it}$, $-\pi/2 < t < \pi/2$. **Ans:** π

17. Which of the following integrals are compatible to apply the Cauchy theorem?

1. $\oint_C \frac{sinz}{z+2i}\,dz$, C : $|z|=1$.
2. $\oint_C \frac{sinz}{z+2i}\,dz$, C : $|z+3i|=1$.
3. $\oint_C e^{\bar{z}}\,dz$, C : $|z-3i|=6$.
4. $\oint_{|z|=b} \frac{1}{z^2+bz+1}\,dz$, $0<b<1$.
5. $\oint_{|z|=3} \frac{1}{1-e^z}\,dz$.

6. $\int_0^{1+i} z^3\,dz$, along $y = x$.

1. Evaluate $\oint_{|z|=3} \frac{1}{z^3-z}\,dz$, **Ans:** 0

2. Evaluate $\oint_C \frac{z^2}{z-2}\,dz$ where C is the boundary of a triangle with vertices -1, 0 and $2i$. **Ans:** 0

3. Evaluate the following integrals around the contour prescribed against it
   (i) $\oint_C \frac{z+e^z}{(z+i\pi)^3}dz$; C: $z = 7e^{it}$, $0 \le t \le 2\pi$. **Ans:** $-\pi i$.
   (ii) $\oint_C \frac{1}{z^3(z-2)^2}dz$; C: $|z-3| = 2$, **Ans:** $\frac{-3\pi i}{8}$.
   (iii) $\oint_C \frac{z+e^z}{(z+i\pi)^3}dz$; C: $z = 7e^{it}$, $0 \le t \le 2\pi$. **Ans:** $-\pi i$.
   (iv) $\oint_C \frac{1}{z^3(z-2)^3}dz$; C: $|z-1| = 3$, **Ans:** 0.
   (v) $\frac{1}{2\pi i}\oint_{|z|=10} \frac{3z^4}{(z-6i)}dz$. **Ans:** 3×6^4
   (vi) $\oint_{|z-1|=5} \frac{1}{z^4-1}dz$. **Ans:** 0.
   (vii) $\oint_{|z-i|=3/2} \frac{1}{z^4(z+i)}dz$. **Ans:** $-4\pi i$.
   (viii) $\oint_{|z-1|=3} \frac{e^z-z^2}{(z-2)^3}dz$. **Ans:** $2\pi i(2e^4-1)$;
   (ix) $\oint_{|z|=2} \frac{\sin z}{(z-1)^2}dz$. **Ans:** $2\pi i \cos 1$;
   (x) $\oint_{|z+i|=3/2} \frac{\cos z}{(z+3i)^6}dz$. **Ans:** 0;
   (xi) $\oint_C \frac{3}{z^2(z+i)^2}dz$; C: $|z| = 5$, **Ans:** 0.
   (xii) $\oint_{|z|=2a} \frac{e^z}{z^2+a^2}dz$. **Ans:** $\frac{2\pi i}{a}\sin a$;
   (xiii) $\oint_{|z-ia|=a} \frac{e^z}{z^2+a^2}dz$. **Ans:** $\frac{e^{ia}\pi}{a}$
   (xiv) $\oint_C \frac{z+1-e^z}{z(z+3)}dz$; C: $|z-i| = 2$, **Ans:** 0.

4. Evaluate $\oint_C (x^2+iy^2)ds$; C: $|z| = 2$ where s is the arc length. **Ans:** $8\pi(1+i)$

Section-C (Laurent's Expansion)

1. State and prove Laurents series expansion of a function $f(z)$.

2. Find Laurents series expansion about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.
(i). $\frac{e^{2z}}{(z-1)^3}$; $z=1$
Ans: $e^2[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{(z-1)}+\frac{4}{3}+\frac{2(z-1)}{3}+$.....; $z=1$ is a pole of order 3 and Series converges for all $z\neq 1$.
(ii). $(z-3)\sin\frac{1}{(z+2)}$; $z=-2$;
Ans: $1-\frac{5}{z+2}-\frac{1}{6(z+2)^2}+\frac{5}{6(z+2)^3}+\frac{1}{120(z+2)^4}-$.....; $z=-2$ is an essential singularity and Series converges for all $z\neq -2$.
(iii). $\frac{z-sinz}{Z^3}$; $z=0$;

Ans: $\frac{1}{3!}-\frac{z^2}{5!}-\frac{z^4}{7!}$..; $z=0$ is a removable singularity. Series converges for all z.
(iv). $\frac{z}{(z+1)(z+2)}$;$z=-2$;
Ans: $\frac{2}{z+2}+1+(z+2)+(z+2)^2+$..; $z=-2$ a simple pole and Series converges for all z in $0<|z+2|<1$.
(v). $\frac{1}{z^2(z-3)^2}$; $z=3$
Ans: $\frac{1}{9(z-3)^2}-\frac{2}{27(z-3)}+\frac{1}{27}-\frac{(z-3)}{243}+$.....; $z=3$ is a pole of order 2 and Series converges for all z in $0<|z-3|<3$.

3. Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a Laurent' s series valid for expansion (a). $1<|z|<3$,
(b). $|z|>3$, (c) $0<|z+1|<2$, (d) $|z|<1$,
Ans: (a). $\cdots\cdots-\frac{1}{2z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}-\cdots\cdots$;
(b). $\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\cdots\cdots$;
(c). $\frac{1}{2(z+1)}-\frac{1}{4}+\frac{(z+1)}{8}-\frac{1}{16}(z+1)^2+\cdots\cdots$;
(d). $\frac{1}{3}-\frac{4z}{9}+\frac{13}{27}z^2+\cdots\cdots\cdots\cdots$.

4. Expand $f(z)=\frac{z}{(z-1)(2-z)}$ in a Laurents series valid for (a). $1<|z|<2$; (b). $|z|>2$;
(c). $0<|z-2|<1$; (d). $|z|<1$; (e). $|z-1|>1$.
Ans: (a). $1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\cdots\frac{1}{z}+\frac{1}{z^2}+\cdots\cdots$;
(b). $-\frac{1}{2}-\frac{3}{z^2}-\frac{7}{z^3}-\frac{15}{z^4}\cdots\cdots$;
(c). $1-\frac{2}{z-2}-(z-2)+(z-2)^2-(z-2)^3+(z-2)^4\cdots\cdots$;
(d). $-\frac{1}{2}z-\frac{3}{4}z^2-\frac{7}{8}z^3-\frac{15}{19}z^4-\cdots\cdots$.
(e). $-\frac{1}{(z-1)}-\frac{2}{(z-1)^2}-\frac{2}{(z-1)^3}\cdots\cdots$;

5. Expand $f(z)=\frac{1}{z-3}$ in a Laurents series valid for (a). $|z|<3$; (b). $|z|>3$
Ans: (a). (i)$-1/3-1/9z-1/27z^2-1/81z^3-\cdots\cdots$;
(ii) $z^{-1}+3z^{-2}+9z^{-3}+27z^{-4}+\cdots\cdots$.

6. Expand $f(z)=\frac{1}{z(z-2)}$ in a Laurent's series valid for (a). $0<|z|<2$; (b). $|z|>2$
Ans: (a). $-\frac{1}{2z}-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}$; (b). $\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^{n+1}}$;

7. Find Taylor's series expansion about the indicated points for each of the following functions. Give the region of convergence of each series.
   (i). $\frac{z}{e^z+1}$; $z = 0$;
   (ii). $\frac{\sin z}{z^2+4}$; $z = 0$
   Ans: (i) $|z| < \pi$; (ii) $|z| < 2$

Section-D (Singularities and Residues)

1. Discuss the types of each singularity for the following functions:
   (i). $\frac{1}{\cos z - \sin z}$; $z = \pi/4$ is a simple pole.
   (ii) . $\frac{\cot \pi z}{(z-a)^2}$; $z = a$ is a pole of order 2.
   (iii). $\frac{e^z}{z^2+1}$; $z = \pm i$ is a simple pole and $z = \infty$ is an essential singularity.
   (iv). $\frac{z}{(z+1)(z+2)}$; $z = -1, -2$ are simple poles.
   (v). $\frac{e^z}{\sin z}$; $z = k\pi, k \in \mathbb{I}$ is a simple pole.
   (vi). $\arcsin \frac{1}{z}$; $z = 1/n\pi, (n \in \mathbb{I}^+)$ simple pole and $z = 0$ an isolated essential singularity.
   (vii). $\arctan \pi z$; $z = 1/n, (n \in \mathbb{I}^+)$ is simple pole and $z = 0$ isolated essential singularity.
   (viii). $\frac{z - sinz}{z^3}$; $z = 0$ is a removable singularity.
   (ix). $\frac{z+1}{z^2-2z}$; $z = 0, 2$ are simple poles.
   (x). $\frac{z^2}{(z-1)^2(z+2)}$; $z = -2$ is a simple pole and $z = 1$ is a pole of order 2.
   (xi). $\frac{1}{z^4+1}$; $e^{\frac{(i\pi+2k\pi)}{4}}, \forall k = 0, 1, 2, 3, 4$ each is a simple pole.
   (xii). $\frac{ze^{iz}}{z^2+a^2}$; $z = \pm ai$ are simple poles.
   (xiii). $\frac{1-e^{2z}}{z^3}$; $z = 0$, a pole of order 2.
   (xiv). $\frac{e^{2z}}{1+e^z}$; $z = i\pi$ is a simple pole.
   (xv). $\frac{e^{2z}}{1-e^z}$; $z = 0$ is a simple pole.
   (xvi). $\frac{z - sinz}{z^5}$; $z = 0$ is a pole of order 2.
   (xvii). $\frac{1}{z^2(e^z-1)}$; $z = 0$ is a pole of order 3.
   (xviii). $\frac{\sinh z}{z^2}$; $z = 0$ is a simple pole.
   (xix). $\frac{(z^2+1)e^z}{(z-1)^3(z-i)}$; $z = 1$ is a pole of order 3 and $z = i$ is a removable singularity.

2. Examine the nature of singularity for the following functions and find its residue:
   Hint: residue is defined as coefficient of $\frac{1}{(z-z_0)}$ in Laurent's series expansion of a function $f(z)$ about a point z_0. Thus,
   residue for a pole of order m is $\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z))$
   residue for a removable singularity=0.
   residue for an essential singularity z_0 can be find by finding coefficient of $\frac{1}{z-z_0}$ in its Laurent's series expansion about a point z_0.
   (i). $\frac{e^z + \sin z}{z^4}$; $z = 0$ is a removable singularity, Residue= 0
   (ii). $\frac{1}{(z+i)^3}$; $z = i$ is a a pole of order 3. Residue = $\frac{3}{6i}$.
   (iii). $\frac{ze^{iz}}{z^2+a^2}$; $z = \pm ai$ are simple poles. Residues are = $\frac{e^{-a}}{2}$. and= $\frac{e^a}{2}$.
   (iv). $\frac{1}{z^2 \sinh z}$; $z = 0$ a pole of order 3. Residue= $-\frac{1}{6}$

3. Using Residue theorem evaluate the following integrals:
   (i). $\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta$, **Ans:** $\frac{2\pi}{\sqrt{3}}$.
   (ii). $\int_0^{2\pi} \frac{1}{1-2a\cos\theta+a^2} d\theta$, $a > 1$ **Ans:** ....
   (iii). $\int_0^{\pi} \frac{1+2\cos\theta}{5+3\cos\theta} d\theta$, **Ans:** $\frac{\pi}{24}$.
   (iv). $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$, **Ans:** $\frac{\pi}{6}$
   (v). $\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta} d\theta$, **Ans:** $\frac{3\pi}{8}$

Unit-V

Prepared by: Dr. Sushil Shukla

Topic: Linear Algebra

Vector space over real numbers/complex numbers, Subspaces, Linear span, Linearly independence, Dimensions and Basis, Linear transformations, Matrix of Linear transformation, Rank-Nullity theorem (Statement and their applications).

Section-A (Vector space over real numbers/complex numbers)

1. **Problem:** Define vector space over real numbers/complex numbers.
   - Hint: A (real) vector space is a collection V of vectors together with two binary operations, addition of vectors (+) and scalar multiplication of a vector by a real number (·), satisfying the following axioms:
     - Let $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be any vectors in V and α, α_1,α_2 be any (real number) scalars.
     - Note: The statement that + and · are binary operations means that $\mathbf{v}_1 + \mathbf{v}_2$ and $\alpha \cdot \mathbf{v}$ are always de ned; that is, they are both vectors in V .
     - [A1] Addition is commutative: $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$.
     - [A2] Addition is associative: $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$.
     - [A3] There exists a zero vector $\mathbf{0}$, with $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
     - [A4] Every vector $\mathbf{v}$ has an additive inverse $-\mathbf{v}$, with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
     - [M1] Scalar multiplication is consistent with regular multiplication: $\alpha_1 \cdot (\alpha_2 \cdot \mathbf{v}) = (\alpha_1\alpha_2) \cdot \mathbf{v}$.
     - [M2] Addition of scalars distributes: $(\alpha_1 + \alpha_2) \cdot \mathbf{v} = \alpha_1 \cdot \mathbf{v} + \alpha_2 \cdot \mathbf{v}$.
     - [M3] Addition of vectors distributes: $\alpha \cdot (\mathbf{v}_1 + \mathbf{v}_2) = \alpha \cdot \mathbf{v}_1 + \alpha \cdot \mathbf{v}_2$.
     - [M4] The scalar 1 acts like the identity on vectors: $1 \cdot \mathbf{v} = \mathbf{v}$.
   - Remark: One may also consider vector spaces where the collection of scalars is something other than the real numbers: for example, there exists an equally important notion of a complex vector space, whose scalars are the complex numbers. (The axioms are the same, except we allow the scalars to be complex numbers.)

2. **Problem:** Prove the following:
   - **a.** The vectors in $\mathbb{R}^n$ are a vector space, for any $n > 0$.
   - **b.** The set of $m \times n$ matrices for any m and any n, forms a vector space.
   - **c.** The complex numbers (the numbers of the form $a + bi$ for real a and b, and where $i^2 = -1$) are a vector space.
   - **d.** The collection of all real-valued functions on any part of the real line is a vector space,

where we de ne the sum of two functions as $(f + g)(x) = f(x) + g(x)$ for every x, and scalar multiplication as $(\alpha \cdot f)(x) = \alpha f(x)$.

e. The zero space with a single element **0**, with $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $\alpha \cdot \mathbf{0} = \mathbf{0}$ for every α, is a vector space.

Section-B (Subspaces)

3. **Problem:** Define Subspace and Show that the set of diagonal 2 × 2 matrices is a subspace of the vector space of all 2 × 2 matrices.
4. **Problem:** Determine whether the set of vectors of the form ⟨t, t, t⟩ forms a subspace of R^3.
   Ans: Yes
   Problem: Determine whether the set of vectors of the form t, t^2 forms a subspace of R^2.
   Ans: No.
5. **Problem:** Determine whether the set of vectors of the form ⟨s, t, 0⟩ forms a subspace of R^3.
   Ans: Yes
6. **Problem:** The collection of solution vectors $\langle x_1, \cdots, x_n \rangle$ to any homogeneous system of linear equations forms a subspace of R^n.

   Ans: Yes
7. **Problem:** Determine whether the set of 2 × 2 matrices of determinant zero is a subspace of the space of all 2 × 2 matrices.
   Ans: No
8. **Problem:** Determine whether the set of 2 × 2 matrices of trace zero is a subspace of the space of all 2 × 2 matrices.
   Ans: Yes
9. **Problem:** Determine whether the set of vectors of the form $\langle x, y, z \rangle$ where $x, y, z \geq 0$ forms a subspace of R^3.
   Ans: No
10. **Problem:** Determine whether the set of vectors of the form $\langle x, y, z \rangle$ where $2x - y + z = 0$ forms a subspace of R^3.
    Ans: Yes
11. **Problem:** The collection of continuous functions on $[a, b]$ is a subspace of the space of all functions on $[a, b]$.

    Ans: Yes
12. **Problem:** The collection of n-times differentiable functions on $[a, b]$ is a subspace of the space of continuous functions on $[a, b]$, for any positive integer n.

    Ans: Yes
13. **Problem:** The collection of all polynomials is a vector space.
    Ans: Yes
14. **Problem:** The collection of solutions to the (homogeneous, linear) differential equation $y'' + 6y' + 5y = 0$ form a vector space.
    Ans: Yes

Section-C (Linear span)

15. **Problem**: Determine whether the vectors $\langle 2, 3, 3\rangle$ and $\langle 4, -1, 3\rangle$ are in span (**v**, **w**), where **v** = $\langle 1, -1, 2\rangle$ and **w** = $\langle 2, 1, -1\rangle$.

    Ans: $\langle 2, 3, 3\rangle$ is not in span and $\langle 4, -1, 3\rangle$ is in span.

16. **Problem**: Show that the vectors $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$, and $\langle 0, 0, 1\rangle$ span R^3.

17. **Problem**: Determine whether the vectors $\langle 1, 2\rangle$, $\langle 2, 4\rangle$, $\langle 3, 1\rangle$ span R^2.
    Ans: Yes

18. **Problem**: Determine whether the vectors $\langle 1, -1, 3\rangle$, $\langle 2, 2, -1\rangle$, $\langle 3, 4, 7\rangle$ span R^3.

    Ans: No

Section-D (Linearly independence)

19. **Problem**: Determine whether the vectors $\langle 1, 1, 0\rangle$, $\langle 0, 2, 1\rangle$ in R^3 are linearly dependent or linearly independent.

    Ans: linearly independent

20. **Problem**: Determine whether the vectors $\langle 1, 1, 0\rangle$, $\langle 2, 2, 0\rangle$ in R^3 are linearly dependent or linearly independent.
    Ans: linearly dependent

21. **Problem**: Determine whether the vectors $\langle 1, 0, 2, 2\rangle$, $\langle 2, -2, 3, 0\rangle$, $\langle 0, 3, 1, 3\rangle$, and $\langle 0, 4, 1, 2\rangle$ in R^4 are linearly dependent or linearly independent.
    Ans: linearly dependent

22. **Problem**: Show that the functions e^x and e^{2x} are linearly independent in the vector space of all real-valued functions.

23. **Problem**: Show that the functions $\sin(x)$ and $\cos(x)$ are linearly independent using the Wronskian.

24. **Problem**: Determine whether the functions $1 + x$, $2 - x$, and $3 + 4x$ are linearly dependent or linearly independent.
    Ans: linearly dependent

Section-E (Dimensions and Basis)

25. **Problem**: Show that the vectors $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$, $\langle 0, 0, 1\rangle$ form a basis for R^3.

26. **Problem**: Show that the vectors $\langle 1, 1, 1\rangle$, $\langle 2, -1, 1\rangle$, $\langle 1, 2, 1\rangle$ form a basis for R^3.

27. **Problem**: Show that the vectors $\langle 1, 1, 0\rangle$ and $\langle 1, 1, 1\rangle$ are not a basis for R^3.

28. **Problem**: Show that the vectors $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$, $\langle 0, 0, 1\rangle$, $\langle 1, 1, 1\rangle$ are not a basis for R^3.

29. **Problem**: Find a basis for the space W of polynomials $p(x)$ of degree ≤ 3 such that $p(1) = 0$.

Ans: $\left(x^3 - 1, x^2 - 1, x - 1\right)$

30. **Problem:** Determine whether the vectors $\langle 1, 2, 1\rangle$, $\langle 2, -1, 2\rangle$, $\langle 3, 3, 1\rangle$ form a basis of R^3.
    Ans: Yes

31. **Problem:** Show that the dimension of R^n is n.
    Hint: Since the n standard unit vectors form a basis.

Section-F (Linear transformations, Matrix of Linear transformation)

32. **Problem:** If $V = W = \mathbb{R}^2$, show that the map T which sends $\langle x, y\rangle$ to $\langle x, x + y\rangle$ is a linear transformation from V to W.

33. **Problem:** If $V = M_{2\times 2}(\mathbb{R})$ and $W = \mathbb{R}$, determine whether the trace map is a linear transformation from V to W.

    Ans: Yes

34. **Problem:** If $V = M_{2\times 2}(\mathbb{R})$ and $W = \mathbb{R}$, determine whether the determinant map is a linear transformation from V to W.

    Ans: No

35. **Problem:** If V is the vector space of all differentiable functions and W is the vector space of all functions, determine whether the derivative map D sending a function to its derivative is a linear transformation from V to W.

    Ans: Yes

36. **Problem:** Show that if $V = W = \mathbb{R}^2$, then the map T which sends $\langle x, y\rangle$ to $\langle ax + by, cx + dy\rangle$ for any a, b, c, d is a linear transformation.
37. **Problem:** Show that if $V = \mathbb{R}^m$ (thought of as $m \times 1$ matrices) and $W = \mathbb{R}^n$ (thought of as $n \times 1$ matrices) and A is any $n \times m$ matrix, then the map T sending $\mathbf{v}$ to $A\mathbf{v}$ is a linear transformation.

Section-G (Rank-Nullity theorem (Statement and their applications))

38. **Problem:** If $T: \mathbb{R}^2 \to \mathbb{R}^3$ is the linear transformation with $T(x, y) = (x + y, 0, 2x + 2y)$, then find a basis for the kernel and for the image of T.
    Ans: Basis for the kernel is given by the single vector (1, -1) and a basis for the image is given by the single vector (1,0,2).
39. **Problem:** If $T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ is the trace map, find the nullity and the rank of T and verify the nullity-rank theorem.

    Ans: Nullity is 3 and rank is 1.

40. **Problem:** True or false

    (a) There is a 3×6 matrix whose kernel is two-dimensional. Ans: False

    (a) If A is a 4×3 matrix and the equation $A\vec{x} = \vec{0}$ has no nonzero solutions, then $\operatorname{rank} A = 3$.

    Ans: True